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Semidirect products and the Pukanszky condition

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Abstract

We study the general geometrical structure of the coadjoint orbits of a semidirect product formed by a Lie group and a representation of this group on a vector space. The use of symplectic induction methods gives new insight into the structure of these orbits. In fact, each coadjoint orbit of such a group is obtained by symplectic induction on some coadjoint orbit of a "smaller" Lie group. We study also a special class of polarizations related to a semidirect product and the validity of Pukanszky's condition for these polarizations. Some examples of physical interest are discussed using the previous methods. © 1998 Elsevier Science B.V.

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1. Introduction

Polarized coadjoint orbits of a Lie group G, are good candidates for geometrically quantized phase spaces. They also play a central role in representation theory, more specifically in the context of Kirillov's "orbit method". In the case of exponential groups, Pukanszky showed [10] that the orbit method leads to irreducible unitary representations of G if and only if the polarization satisfies a certain condition, known as Pukanszky's condition (see Lemma 6.2 (1) with $\mathfrak{d} = \mathfrak{e}$). This method has been adapted to the case of complex polarizations, especially for solvable Lie groups: Auslander and Kostant [7] showed that Pukanszky's condition was needed in order to guarantee the irreducibility of the representations obtained via holomorphic induction from the real polarizing subgroup $D \subset G$ (see below).

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In its initial formulation, Pukanszky's condition means that the coadjoint orbit in question contains an affine plane, constructed out by the polarization. Only recently [2,3], it has been realized that validity of Pukanszky's condition is equivalent to the fact that the corresponding coadjoint orbit is symplectomorphic to a modified cotangent bundle, obtained by symplectic induction from a point. The physical consequences of this symplectomorphism have been studied in the previous references for the coadjoint orbits of the Poincaré group, which is a semidirect product.

Our aim is to give, on the one hand, a detailed analysis of the geometrical structure of the semidirect product coadjoint orbits, for the case where this product is formed by a Lie group K and a representation $\rho : K \to GL(V)$ on a vector space V [11]. On the other hand, we consider this very interesting geometrical structure in the framework of Pukanszky's condition. Summarizing the results of this article, we mention the following three points:

- the coadjoint orbits of a semidirect product present several analogies with the cotangent bundles and under certain conditions they are in reality cotangent bundles of K-orbits in the dual V^* of the vector space V;
- the validity of Pukanszky's condition for a special class of polarizations of the semidirect product $G = K \times_\rho V$ is equivalent to the validity of the same condition for "smaller" polarizations associated to the homogeneous part K;
- the coadjoint orbits of the semidirect product $G = K \times_\rho V$ are obtained by symplectic induction on coadjoint orbits of appropriate subgroups of the homogenous part K.

In what concerns the third point, a variant of the symplectic induction we are using here gave recently the same result [8]; see also [17] for a more general treatment in the context of symplectic Mackey's theory.

We discuss finally three examples of semidirect product and we apply the previous formalism in the geometrical study of their coadjoint orbits. The semidirect products in question are the special Euclidean, the Galilei and the Bargmann group, three Lie groups whose coadjoint orbits are, respectively, related to geometrical optics, polarization of light and to the dynamics of non-relativistic particles.

2. The semidirect product

In this section, we fix the notation concerning the semidirect product following [11].

Consider a Lie group K with Lie algebra $\mathfrak{k}$; let $(\kappa, f) \mapsto \kappa \cdot f$ be the coadjoint representation of K on $\mathfrak{k}^*$, the dual of the Lie algebra $\mathfrak{k}$, and $(A, f) \mapsto A \cdot f$ its derivative, $\kappa \in K, A \in \mathfrak{k}, f \in \mathfrak{k}^*$. If $\rho : K \to GL(V)$ is a representation of K on the vector space V, then we note $\rho(\kappa)v = \kappa \cdot v$, $\kappa \in K$, $v \in V$. We note accordingly by $(\kappa, p) \mapsto \kappa \cdot p$ the contragradient representation of ρ, $p \in V^*$ and by $(A, v) \mapsto A \cdot v$ and $(A, p) \mapsto A \cdot p$ the corresponding derivative representations of $\mathfrak{k}$ on V and V^*.

We form now the semidirect product $G = K \times_\rho V$. As a set $G = K \times V$ and the group operation in G is given by

$$(\kappa, v)\cdot(\lambda, u) = (\kappa\lambda, \kappa\cdot u + v) \quad \forall(\kappa, v), (\lambda, u)\in G. \tag{2.1}$$

When the representation ρ is understood, we write simply $G = K \rtimes V$.

The Lie algebra of this group is $\mathfrak{g} = \mathfrak{k}\oplus V$ (as a vector space) and the Lie algebra structure is given by the bracket

$$[(A, a), (B, b)] = ([A, B], A\cdot b - B\cdot a) \quad \forall(A, a), (B, b)\in\mathfrak{g}. \tag{2.2}$$

We will note $\mathfrak{g} = \mathfrak{k}\oplus_\rho V$.

By identifying the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{k}^*\oplus V^*$, we can express the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ as

$$\mu(\xi) = f(A) + p(a) \quad \forall\mu = (f, p)\in\mathfrak{g}^*, \quad \xi = (A, a)\in\mathfrak{g} \tag{2.3}$$

and the adjoint and coadjoint representations of G on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, by the following relations:

$$\mathrm{Ad}(\kappa, v)(A, a) = (\mathrm{Ad}(\kappa)A, \kappa\cdot a - \rho'(\mathrm{Ad}(\kappa)A)v) \quad \forall(\kappa, v)\in G, \quad (A, a)\in\mathfrak{g}, \tag{2.4}$$

$$\mathrm{Coad}(\kappa, v)(f, p) = (\kappa\cdot f + \kappa\cdot p\odot v, \kappa\cdot p) \quad \forall(\kappa, v)\in G, \quad (f, p)\in\mathfrak{g}^*, \tag{2.5}$$

where $p\odot v$ is the element of $\mathfrak{k}^*$ defined by

$$(p\odot v)(A) = p(A\cdot v) = -(A\cdot p)(v) \quad \forall A\in\mathfrak{k}, \quad p\in V^*, \quad v\in V. \tag{2.6}$$

For $p\in V^*$ we denote by K_p the isotropy subgroup of p formed by those $\kappa\in K$ such that $\kappa\cdot p = p$. It is clear that the Lie algebra of K_p is given by the vector space $\mathfrak{k}_p = \{A\in\mathfrak{k} \mid A\cdot p = 0\}$. Then if we define the linear map $\tau_p : \mathfrak{k}\to V^*$ by

$$\tau_p(A) = -A\cdot p \quad \forall A\in\mathfrak{k}, \tag{2.7}$$

we have the equality $\mathfrak{k}_p = \ker\tau_p$.

We express now the element $p\odot v\in\mathfrak{k}^*$ in terms of the map τ_p. The dual $\tau_p^* : V\to\mathfrak{k}^*$ of τ_p is given by the relation $\tau_p^*(v)(A) = \tau_p(A)(v) = -(A\cdot p)(v)$, and so $\tau_p^*(v) = p\odot v$, $\forall p\in V^*, \forall v\in V$.

Let now $\mathfrak{k}_p^\circ$ be the annihilator of $\mathfrak{k}_p$; then if $i_p^* : \mathfrak{k}^*\to\mathfrak{k}_p^*$ is the projection, $i_p : \mathfrak{k}_p \hookrightarrow \mathfrak{k}$, we have $\mathfrak{k}_p^\circ = \ker i_p^*$. The following is a useful lemma from [11], giving a characterization of the annihilator $\mathfrak{k}_p^\circ$ in terms of the linear map τ_p.

Lemma 2.1. $\mathfrak{k}_p^\circ = \mathrm{im}\,\tau_p^*$.

Proof. We give here a different and simple proof using Lagrange multipliers. Indeed, we have $\mathfrak{k}_p = \tau_p^{-1}(0)\subset\mathfrak{k}$ and the element $A\in\mathfrak{k}_p$ is a critical point of the map $i_p^* f : \mathfrak{k}_p\to\mathbb{R}$, for $f\in C^\infty(\mathfrak{k})$, if and only if there exists an element $v\in V^{**}\cong V$ such that A is a critical point of $f - v\circ\tau_p$. Choosing $f\in\mathfrak{k}^*$ (and therefore linear), we find that $f\in\mathfrak{k}_p^\circ$ if and only if $\exists v\in V$: $f = v\circ\tau_p = p\odot v$. □

3. Orbits – isotropy subgroups

We recall now the structure of the coadjoint orbits of a semidirect product studied by Rawnsley [11], and we exploit in more detail the structure of the isotropy subgroups with respect to the coadjoint representation for the semidirect product. According to [11], the coadjoint orbits of a semidirect product are classified by the coadjoint orbits of "little" groups, which are isotropy subgroups of its homogeneous part (see also [4]). In fact, fibre bundles having these coadjoint orbits as fibres, completely characterize the coadjoint orbits of the semidirect product. As we shall see later on (Section 10), the little-group coadjoint orbits play an even deeper role for the geometrical structure of the corresponding semidirect product coadjoint orbit.

Let now $Z = \mathbb{O}_p^K$, $p \in V^*$ be an orbit of K in V^* with respect to the representation ρ^*. A bundle of little-group orbits over Z is a fibre bundle $\pi : Y \to Z$ such that each fibre $Y_p = \pi^{-1}(p)$ be a coadjoint orbit of the isotropy subgroup K_p.

We construct the bundle of little-group orbits as follows. Consider elements $p \in V^*$, $\phi \in \mathfrak{k}_p^*$ and let Z, Y_p be the corresponding orbits under the actions of K and K_p, respectively, $Z = \mathbb{O}_p^K$, $Y_p = \mathbb{O}_\phi^{K_p}$. There is a left action of the isotropy subgroup $K_p \subset K$ on the product $K \times Y_p$ given by

$$h \cdot (\kappa, \phi) = (\kappa h^{-1}, h \cdot \phi). \tag{3.1}$$

We define the bundle of little-group orbits Y as the quotient $Y = (K \times Y_p)/K_p$, i.e., Y is the fibre bundle associated to the principal bundle $K \to Z$ with respect to the coadjoint action of K_p on Y_p. The group K acts on Y as follows. If $\psi \in Y_p$, then for $\kappa \in K$ we define the point $\kappa \cdot \psi \in Y_{\kappa \cdot p}$ as

$$(\kappa \cdot \psi)(A) = \psi(\mathrm{Ad}(\kappa^{-1})A) \quad \forall A \in \mathfrak{k}_{\kappa \cdot p}. \tag{3.2}$$

Consequently, the following choice to represent the points of Y (K_p-orbits in $K \times Y_p$) is appropriate: $K_p \cdot (\kappa, \phi) = \kappa \cdot \phi \in Y_{\kappa \cdot p}$. We can now define the projection $\pi : Y \to Z$ by $\pi(K_p \cdot (\kappa, \phi)) = \kappa \cdot p \in Z$.

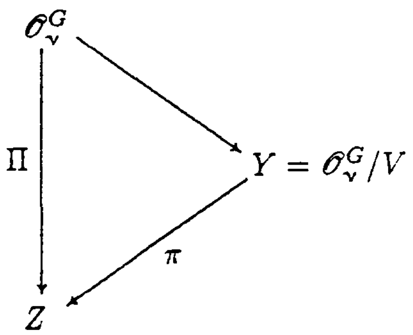
It is easy to verify that this construction is independent of the point $p \in Z$. Then the following proposition [11] clarifies the role of the bundles of little-group orbits; see also Section 10.

Proposition 3.1. *There is a bijection between the set of bundles of little-group orbits and the set of coadjoint orbits of G on $\mathfrak{g}^*$.*

Consider now the coadjoint orbit $\mathbb{O}_\nu^G$ of $\nu = (f, p)$ in $\mathfrak{g}^*$ and the corresponding fibre bundle of little-group orbits Y. This orbit is fibred over the K-orbit Z of the point $p \in V^*$ and the fibre is of the form $(\mathbb{O}_\nu^G)_q = K_q \cdot h + q \odot V$, $q = \kappa \cdot p \in Z$, $h = \kappa \cdot f$ for $\kappa \in K$ (relation (2.5)). Thus if $y \in (\mathbb{O}_\nu^G)_q$, then there exists an element $(\lambda, v) \in K_q \times_\rho V$ such that $y = \lambda \cdot h + q \odot v$ and so $i_q^*(y) = i_q^*(\lambda \cdot h + q \odot v) = i_q^*(\lambda \cdot h) = \lambda \cdot i_q^* h \in Y_q$. It is clear that the projection $i_q^* : \mathfrak{k}^* \to \mathfrak{k}_q^*$ defines in fact a projection $i_q^* : (\mathbb{O}_\nu^G)_q \to Y_q$ between the fibres of $\mathcal{O}_\nu^G$ and Y. Furthermore, the fibre $(i_q^*)^{-1}(\varphi)$ for $\varphi \in Y_q$, is the orbit of the point $q \in V^*$ under the action of the linear subgroup $V \subset G$. We have thus proved the lemma [11]:

Lemma 3.2. *The coadjoint orbit $\mathcal{O}_\nu^G$ of the element $\nu = (f, p) \in \mathfrak{g}^* = \mathfrak{k}^* \oplus V^*$ is a fibre bundle over the bundle of little-group orbits whose typical fibre is the orbit of $p \in V^*$ under the action of the subgroup $V \subset G$.*

We summarize the previous results in the commutative diagram



where $\Pi : \mathcal{O}_\nu^G \to Z$ is the projection.

We study finally the isotropy subgroup G_ν of the point $\nu = (f, p) \in \mathfrak{g}^*$ with respect to the coadjoint action. Let $\phi = i_p^* f$ and $(K_p)_\phi$ be the isotropy subgroup of $\phi \in \mathfrak{k}_p^*$ with respect to the coadjoint action of K_p on $\mathfrak{k}_p^*$. If $(\kappa, v) \in G_\nu$, then $\kappa \cdot p = p$ and $\kappa \cdot f + p \odot v = f$ which means that $\kappa \cdot \phi = \phi \Rightarrow \kappa \in (K_p)_\phi$. We have thus an epimorphism $j : G_\nu \to (K_p)_\phi$ given by $j(\kappa, v) = \kappa$. The kernel of j is calculated easily: $\ker j = \{(\kappa, v) \in G_\nu \mid j(\kappa, v) = e\} = \{(e, v) \in G_\nu\}$. But the element (e, v) belonging to G_ν is such that $e \cdot f + p \odot v = f \Rightarrow p \odot v = 0$, thus $v \in \ker \tau_p^*$. On the other hand, $\ker \tau_p^*$ is a vector subgroup of G_ν as the inclusion map $i : \ker \tau_p^* \to G_\nu$ given by $i(v) = (e, v)$ indicates.

We conclude that $\ker j = \ker \tau_p^*$ and we have the following exact sequence:

$$0 \longrightarrow \ker \tau_p^* \xrightarrow{\ i\ } G_\nu \xrightarrow{\ j\ } (K_p)_\phi \longrightarrow e, \tag{3.3}$$

which gives us all the possible information about the structure of the isotropy subgroup G_ν.

We note that G_ν is not in general equal to the semidirect product of $(K_p)_\phi$ and $\ker \tau_p^*$ because $(K_p)_\phi$ is not in general a subgroup of G_ν. In fact, if it were, we would have $(\kappa, 0) \in G_\nu$ for each $\kappa \in (K_p)_\phi$. But in such a case we find $\kappa \cdot f + p \odot 0 = f \Rightarrow \kappa \in K_f$, so $(K_p)_\phi \subset K_f$; clearly this condition is not in general satisfied.

Conversely now, the inclusion $(K_p)_\phi \subset K_f$ induces a group monomorphism $m : (K_p)_\phi \to G_\nu$ given by $m(\kappa) = (\kappa, 0)$ for in that case we have $\kappa \cdot f + p \odot 0 = f$ for each $\kappa \in (K_p)_\phi$. The following lemma is thus proved.

Lemma 3.3. *The inclusion $(K_p)_\phi \subset K_f$ is a necessary and sufficient condition for the isotropy subgroup G_ν to be the semidirect product $(K_p)_\phi \times_\rho \ker \tau_p^*$.*

4. Submanifolds – symplectic structure

The coadjoint orbits of a semidirect product $G = K \times_\rho V$ possess always two natural submanifolds: the K-orbit $L = \mathcal{O}_\nu^K$ of $\nu \in \mathfrak{g}^*$ and the V-orbit $N = \mathcal{O}_\nu^V$ of the same element. We observe that in the case where $K_p \subset K_f$, $\nu = (f, p) \in \mathfrak{g}^*$, the orbits L and Z are diffeomorphic: $Z = K/K_p$ and $L = K/(K_p \cap K_f) = K/K_p$. In this section we will study the submanifolds L and N as well as the symplectic structure of the coadjoint orbit $\mathcal{O}_\nu^G$.

Convention 4.1. If ξ is an element of a Lie algebra $\mathfrak{g}$, then we denote by $\xi_{\mathfrak{g}^*}$ the fundamental vector field of the coadjoint action on $\mathfrak{g}^*$:

$$(\xi_{\mathfrak{g}^*})_\mu = \xi \cdot \mu \quad \forall \mu \in \mathfrak{g}^*. \tag{4.1}$$

Also, the symplectic structure we are using on a coadjoint orbit in $\mathfrak{g}^*$, is given by:

$$\omega_\mu((\xi_{\mathfrak{g}^*})_\mu, (\eta_{\mathfrak{g}^*})_\mu) = -\mu([\xi, \eta]), \tag{4.2}$$

where μ is a point on the orbit.

For the case of the semidirect product in which we are interested, Eq. (4.2) takes the following form: if $\xi = (A, a)$ and $\eta = (B, b)$, $A, B \in \mathfrak{k}$, $a, b \in V$, then

$$\omega_\mu((\xi_{\mathfrak{g}^*})_\mu, (\eta_{\mathfrak{g}^*})_\mu) = (A \cdot h + q \odot a)(B) + \tau_q(A)(b) \tag{4.3}$$

if $\mu = (h, q) \in \mathfrak{g}^*$.

We find first for $o \in L$ and $n \in N$, the tangent spaces T_oL and T_nN explicitly. Using the fact that L and N are homogeneous spaces of the groups K and V, respectively, we have

$$T_oL = \{(A \cdot h, A \cdot q) \mid A \in \mathfrak{k}\}, \quad o = (h, q) = (\kappa, 0) \cdot \nu \in L \tag{4.4}$$

and

$$T_nN = \{(p \odot u, 0) \mid u \in V\} \cong \operatorname{im} \tau_p^* = \mathfrak{k}_p^\circ, \quad n = (e, v) \cdot \nu \in N. \tag{4.5}$$

We observe here that the tangent space T_nN does not depend on the point $n \in N$, in accordance with the affine plane structure of $N = \{(f + p \odot v, p) \mid v \in V\} = \nu + \operatorname{im} \tau_p^* \times \{0\}$.

In order to obtain now a characterization for the submanifolds L and N, we search for the orthogonal complements of the tangent spaces T_oL and T_nN. We find easily, using relations (4.3) and (4.4)

$$(T_oL)^\perp = \{(0, B \cdot q) \in T_o\mathcal{O}_\nu^G \mid B \in \mathfrak{k},\ B \cdot h \in q \odot V\}. \tag{4.6}$$

It is clear that generally, the orthogonal complement $(T_oL)^\perp$ has no relation to the tangent space T_oL. But in the case where f defines a cohomology class, $[f] \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$, we have $B \cdot f = 0$ hence $B \cdot h = 0$, $\forall B \in \mathfrak{k}$, which implies, by (4.6) and (4.4), that

$$(T_oL)^\perp = T_oL.$$

Thus the condition $[f] \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$ means that the submanifold L is Lagrangian.

We turn to the case of $(T_nN)^\perp$. Easy calculation shows that

$$(T_nN)^\perp = \{(B \cdot (f + p \odot v) + p \odot b, 0) \mid B \in \mathfrak{k}_p,\ b \in V\},$$

which clearly leads to the inclusion $T_nN \subset (T_nN)^\perp$. Thus the submanifold N is always an isotropic submanifold of the coadjoint orbit $\mathcal{O}_\nu^G$.

When $K_p \subset K_f$, N becomes a Lagrangian submanifold. Indeed, the isomorphism $T_nN \cong \mathfrak{k}_p^\circ$ implies that the dimensions of N and Z are equal; furthermore, if $K_p \subset K_f$, then the tangent spaces of N and L at ν are complementary, so $\dim N + \dim L = \dim \mathcal{O}_\nu^G$ and $Z \cong L \Rightarrow \dim Z = \dim L$; finally $2 \dim N = \dim \mathcal{O}_\nu^G$.

We have now a useful characterization of the cotangent space T_q^*Z, $q \in Z$:

Lemma 4.2. *For each point $q \in Z$, the cotangent space T_q^*Z is naturally isomorphic to the quotient $V/\ker \tau_q^*$.*

Proof. It follows directly from the isomorphism $T_qZ \cong \mathfrak{k}/\mathfrak{k}_q$ and Lemma 2.1. □

Using the previous lemma, one can investigate further the consequences of the condition $K_p \subset K_f$ on the structure of the coadjoint orbit $\mathcal{O}_\nu^G$, where $\nu = (f, p) \in \mathfrak{g}^*$. In fact, let $\Pi : \mathcal{O}_\nu^G \to Z$ be the projection and $q \in Z$; then, by Eq. (2.5), one easily finds that for $\mu = (h, q) \in \mathcal{O}_\nu^G$, the fibre $\Pi^{-1}(\Pi(\mu))$ is of the form

$$\Pi^{-1}(q) = (h, q) + q \odot V \times \{0\}. \tag{4.7}$$

Lemma 4.2 applied to the case $K_p \subset K_f$, makes clear that $\mathcal{O}_\nu^G$ is isomorphic (as a manifold) to the cotangent bundle T^*Z when $K_p \subset K_f$.

We make now the following remark concerning the bundle Y of little-group orbits, under the condition $K_p \subset K_f$. If $\phi = i_p^* f$ (notation of Sections 2 and 3), the typical fibre of Y is $Y_p = \mathcal{O}_\phi^{K_p}$. It is immediate that for each $\kappa \in K_p$, $\kappa \cdot \phi = i_p^*(\kappa \cdot f) = \phi$ which implies that $Y_p = \{\phi\}$; thus the fibre bundle Y and the orbit Z are isomorphic as manifolds.

We study finally the case $[f] \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$ and its consequences on the structure of the coadjoint orbit $\mathcal{O}_\nu^G$. We use the following well-known property of the coadjoint action:

Property 4.3. *If $f \in \mathfrak{k}^*$ defines a cohomology class, $[f] \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$, then the coadjoint orbit $Q = \mathcal{O}_f^K$ is a manifold of dimension zero.*

In other words, the isotropy subgroup K_f is at the same time open and closed in K; thus, using the previous discussion and relation (2.5) we find that the fibre bundle $\Pi : \mathcal{O}_\nu^G \to Z$ defines a covering space of the cotangent bundle T^*Z. In particular, the bundle of little-group orbits is a covering space of the orbit Z.

Let us summarize with the proposition:

Proposition 4.4. *The coadjoint orbit $\mathcal{O}_\nu^G$ of a semidirect product $G = K \times_\rho V$, $\nu = (f, p) \in \mathfrak{g}^*$, possesses always two natural submanifolds L and N which have transversal intersection at $\nu \in L \cap N$: L is the orbit of ν under the action of $K \subset G$ and N the orbit of* *the same point under $V \subset G$. N is always an isotropic submanifold of the coadjoint orbit $\mathcal{O}^G_\nu$.*

1. *Suppose that $K_p \subset K_f$; then*
   (a) *the orbit Z is diffeomorphic to L and N is a Lagrangian submanifold;*
   (b) *the coadjoint orbit $\mathcal{O}^G_\nu$ is diffeomorphic to the cotangent bundle T^*Z;*
   (c) *the bundle of little-group orbits Y and the orbit Z are identical.*
2. *Suppose that f defines a cohomology class, $[f] \in \mathrm{H}^1(\mathfrak{k}, \mathbb{R})$; then*
   (a) *L and N are Lagrangian submanifolds of $\mathcal{O}^G_\nu$;*
   (b) *the orbit $\mathcal{O}^G_\nu$ is a covering space of the cotangent bundle T^*Z;*
   (c) *the bundle of little-group orbits Y is a covering space of the orbit Z.*

More generally, we can define a foliation $\mathcal{F}$ on the coadjoint orbit $\mathcal{O}^G_\nu$ in the following way: if $o = (h, q) = (\kappa \cdot f, \kappa \cdot p) \in L$, $\kappa \in K$, then we choose the leaf $\mathcal{F}_o$ as $\mathcal{F}_o = \mathcal{O}^V_o = \{(h + q \odot v, q) \mid v \in V\}$. Then, using the techniques of Proposition 4.4, one easily proves:

Proposition 4.5. *The coadjoint orbit $\mathcal{O}^G_\nu$ of a semidirect product $G = K \times_\rho V$, $\nu = (f, p) \in \mathfrak{g}^*$, possesses an isotropic foliation whose leafs are the affine spaces $\mathcal{F}_o = \mathcal{O}^V_o$, $o \in L$ (see Proposition 4.4). In the case where $K_p \subset K_f$, the foliation $\mathcal{F}$ is Lagrangian.*

In view of Lemma 3.2, the following is immediate:

Corollary 4.6. *The foliation $\mathcal{F}$ of Proposition 4.5 is always regular and the quotient $\mathcal{O}^G_\nu / \mathcal{F}$ is equal to the bundle of little-group orbits.*

We observe here the analogy with the cotangent bundle: in fact, if T^*M is the cotangent bundle of a manifold M, then the fibres $T^*_m M$, $m \in M$, define a Lagrangian foliation of T^*M. We will clarify in what follows this similarity by direct calculation of the symplectic form ω of the coadjoint orbit $\mathcal{O}^G_\nu$ in terms of the symplectic structures of T^*Z and $Q = \mathcal{O}^K_f$.

Fix now an element $\nu = (f, p) \in \mathfrak{g}^*$ and let $\sigma : G \to \mathcal{O}^G_\nu$, $\sigma_1 : G \to Q$ and $\sigma_2 : G \to T^*Z$ be the mappings defined by $\sigma_1(\kappa, v) = \kappa \cdot f$, $\sigma_2(\kappa, v) = (\kappa \cdot p, [v]_{\kappa \cdot p})$ and σ is simply the projection $G \to G/G_\nu$; $[v]_{\kappa \cdot p}$ means the equivalence class of v in the quotient $V / \ker \tau^*_{\kappa \cdot p}$, according to Lemma 4.2.

Theorem 4.7. *The canonical symplectic structures ω, ω_Q and ω_Z of $\mathcal{O}^G_\nu$, Q and T^*Z, respectively, are related by the following equation:*

$$\sigma^*\omega = \sigma_1^*\omega_Q + \sigma_2^*\omega_Z. \tag{4.8}$$

Proof. Let $\Omega = \sigma_1^*\omega_Q + \sigma_2^*\omega_Z \in \Omega^2(G)$; Ω is a closed 2-form. If $\xi = (A, a)$, $\eta = (B, b) \in \mathfrak{g}$, let also ξ^{r} and η^{r} be the corresponding right invariant vector fields on G. Then, easy calculation shows that

$$\Omega(\xi^{\mathrm{r}}, \eta^{\mathrm{r}}) \mid_g = q(B \cdot (A \cdot v + a)) - q(A \cdot (B \cdot v + b)) - h([A, B])$$

if $q = \kappa \cdot p$, $h = \kappa \cdot f$ and $g = (\kappa, v)$. On the other hand, the fact that $\mathcal{O}_\nu^G$ is the quotient G/G_ν implies directly $T_g\sigma(\xi^{\mathrm{r}}(g)) = \xi_{\mathfrak{g}^*}(\mu)$, $\mu = \sigma(g)$. By Convention 4.1 and relation (4.3), this means that $\sigma^*\omega$ is exactly Ω. □

Remark 4.8. It must be emphasized that the previous result is closely related to the choice of a point of $\mathcal{O}_\nu^G$ (here we choose the origin ν of the orbit). In this sense, the splitting of the symplectic structure ω of the coadjoint orbit is not canonical.

5. Coadjoint orbits, modified cotangent bundles and coisotropic embeddings

A cotangent bundle T^*M equipped with the 2-form $\widetilde{\omega}_M = \omega_M + \tau^*\alpha_0$, where $\omega_M = \mathrm{d}\theta_M$ is the canonical symplectic form of T^*M and $\alpha_0 \in \Omega^2(M)$ is such that $\mathrm{d}\alpha_0 = 0$, $\tau : T^*M \to M$, is called modified cotangent bundle. We denote the pair $(T^*M, \omega_M + \tau^*\alpha_0)$ by $T^\sharp M$, when α_0 is understood. The form $\widetilde{\omega}_M$ is always non-degenerate, as one readily verifies, so $T^\sharp M$ is also a symplectic manifold.

This notion is closely related to physical problems. As most important, we mention the phase space of a charged particle in general relativity, in the presence of an external electromagnetic field, e.g., [12,13] and the problem of localization of relativistic particles of mass zero [3]. In this section we will see that it is also useful in the geometry of the coadjoint orbits of a semidirect product.

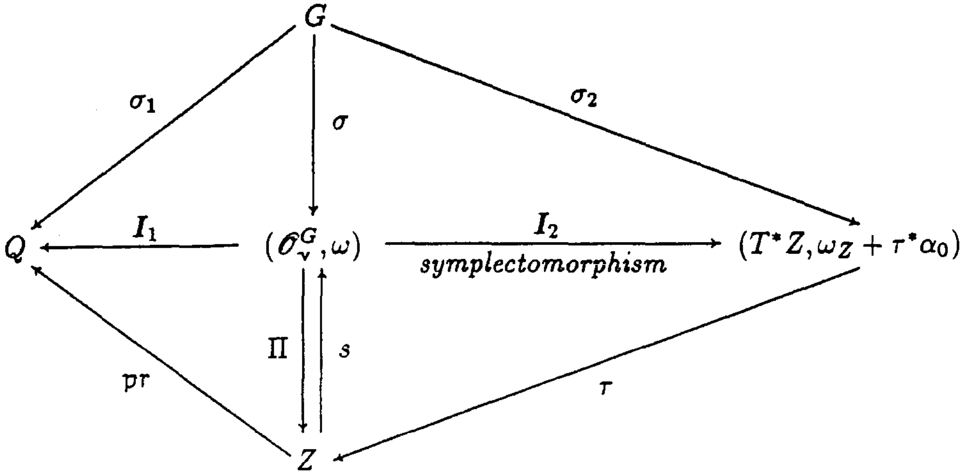
Consider an element $\nu = (f, p) \in \mathfrak{k}^* \oplus V^*$ with $K_p \subset K_f$. Then we know that the orbit Z forms a fibre bundle over Q with typical fibre K_f/K_p, the orbit of p under K_f. In addition, if $\mathbf{pr} : Z \to Q$ is the projection, $\mathbf{pr}(\kappa \cdot p) = \kappa \cdot f$, the 2-form $\alpha_0 = \mathbf{pr}^*\omega_Q \in \Omega^2(Z)$ is a presymplectic structure on Z.

Proposition 5.1. *If the element $\nu = (f, p) \in \mathfrak{g}^*$ is such that $K_p \subset K_f$, then there exists a global section $s : Z \to \mathcal{O}_\nu^G$ of $\Pi : \mathcal{O}_\nu^G \to Z$ such that $s^*\omega = \alpha_0$. Furthermore, if $\tau : T^*Z \to Z$ is the projection, then there exists a symplectomorphism between $(\mathcal{O}_\nu^G, \omega)$ and $T^\sharp Z = (T^*Z, \omega_Z + \tau^*\alpha_0)$. However, this symplectomorphism is not canonical.*

Proof. Define $s : Z \to \mathcal{O}_\nu^G$ by $s(\kappa \cdot p) = (\kappa \cdot f, \kappa \cdot p)$. The map s is well defined because if $\kappa \cdot p = \lambda \cdot p$, then $\lambda = \kappa \cdot \kappa'$ for $\kappa' \in K_p$ and so $s(\lambda \cdot p) = s(\kappa \cdot p)$. Clearly, we have $\Pi \circ s = id$, so s is a section. Its tangent at $q = \kappa \cdot p$ is given by $T_q s(A \cdot q) = (A \cdot h, A \cdot q) = (A, 0)_{\mathfrak{g}^*}(h, q)$ if $h = \kappa \cdot f$; this makes clear that $s^*\omega = \alpha_0$ (see (4.3)).

Now by Proposition 4.4 we have the diffeomorphism $\mathcal{O}_\nu^G \cong T^*Z$. Using the notation of Theorem 4.7 we observe that if $\sigma(\kappa, v) = \sigma(\lambda, u)$, then $\lambda = \kappa \cdot \kappa'$ for $\kappa' \in K_p$ and $v - u \in \ker \tau^*_{\kappa \cdot p}$. This means that there exist two well-defined mappings $\boldsymbol{I}_1 : \mathcal{O}_\nu^G \to Q$ and $\boldsymbol{I}_2 : \mathcal{O}_\nu^G \to T^*Z$ satisfying $\boldsymbol{I}_i \circ \sigma = \sigma_i$, $i = 1, 2$. Explicitly, if μ is the element of $\mathfrak{g}^*$ given by relation (2.5), then $\boldsymbol{I}_1(\mu) = \kappa \cdot f$ and $\boldsymbol{I}_2(\mu) = (\kappa \cdot p, [v]_{\kappa \cdot p})$. As a consequence, relation (4.8) reads $\omega = \boldsymbol{I}_1^*\omega_Q + \boldsymbol{I}_2^*\omega_Z$ because σ^* is a monomorphism. But $\boldsymbol{I}_1$ satisfies also $\boldsymbol{I}_1 = \mathbf{pr} \circ \tau \circ \boldsymbol{I}_2$ which gives $\omega = \boldsymbol{I}_2^*(\omega_Z + \tau^*\alpha_0)$. Finally, it is elementary to verify that $\boldsymbol{I}_2$ is a diffeomorphism between $\mathcal{O}_\nu^G$ and T^*Z; this means that we have a symplectomorphism between $\mathcal{O}_\nu^G$ and $T^\sharp Z$ which is not canonical because it depends on the choice of a point $\nu = (f, p)$ of the coadjoint orbit in question, for which the condition $K_p \subset K_f$ is satisfied. □

We summarize our knowledge on the structure of $\mathcal{O}_\nu^G$ when $K_p \subset K_f$ in the commutative diagram:



But there exist additional properties of the coadjoint orbit $\mathcal{O}_\nu^G$ when $K_p \subset K_f$. More precisely:

Theorem 5.2. *Let $G = K \times_\rho V$ be a semidirect product and $\nu = (f, p) \in \mathfrak{g}^*$ such that $K_p \subset K_f$. Then the reduction of $\mathcal{O}_\nu^G$ by the submanifold L of Proposition* 4.4 *is symplectomorphic to the coadjoint orbit $Q = K/K_f$. Furthermore, the dual E^* of the characteristic distribution on L defines a symplectic manifold which is a vector bundle over Z and the zero section $s_0 : Z \to E^*$ is a coisotropic embedding of the presymplectic manifold Z.*

Proof. The section s is a diffeomorphism between Z and L and therefore defines a presymplectomorphism $s : (Z, \alpha_0) \to (L, i_L^*\omega), i_L : L \to \mathcal{O}_\nu^G$ is the inclusion, because $s^* i_L^* \omega = (i_L \circ s)^*\omega = \alpha_0$. This ensures that the characteristic distribution E of $i_L^*\omega$ on L, $E = TL \cap (TL)^\perp$, is isomorphic to the kernel of α_0 over Z. But if we reduce Z by $\ker \alpha_0$ we obtain the coadjoint orbit K/K_f. Thus reduction of $\mathcal{O}_\nu^G$ by L gives the symplectic manifold Q.

For the case we are studying, one easily finds that if $o = (\kappa \cdot f, \kappa \cdot p) = (h, q) \in L$, then, see (4.6), $T_oL \cap (T_oL)^\perp = \{(0, A \cdot q) \mid A \in \mathfrak{k}_h\}$. Thus, we conclude that the dual E^* of the characteristic distribution E is a vector bundle over Z with typical fibre $(\mathfrak{k}_f/\mathfrak{k}_p)^*$. Consider now a point $x \in E$; let $q = \kappa \cdot p$ be its projection on Z and $h = \kappa \cdot f = \mathbf{pr}(q)$. Then, by construction of E^*, the tangent space T_xE^* admits the decomposition: $T_xE^* = T_hQ \oplus (\mathfrak{k}_h/\mathfrak{k}_q) \oplus (\mathfrak{k}_h/\mathfrak{k}_q)^*$. This makes clear that there exists a symplectic structure ω_{E^*} on E^* given by $(\omega_{E^*})_x(\xi_1 + \alpha_1 + \beta_1, \xi_2 + \alpha_2 + \beta_2) = (\omega_Q)_h(\xi_1, \xi_2) + \beta_1(\alpha_2) - \beta_2(\alpha_1)$,

$\xi_i \in T_h Q$, $\alpha_i \in (\mathfrak{k}_h/\mathfrak{k}_q)$, $\beta_i \in (\mathfrak{k}_h/\mathfrak{k}_q)^*$, $i = 1, 2$. Let now $s_0 : Z \to E^*$ be the zero section and $\mathfrak{X} = s_0(Z)$. Then each tangent vector $v \in T_x\mathfrak{X}$ admits the decomposition $v = \eta + \gamma + 0$, $\eta \in T_h Q$, $\gamma \in (\mathfrak{k}_h/\mathfrak{k}_q)$. Consequently, the orthogonal complement of this tangent space will be given by $(T_x\mathfrak{X})^\perp = \{0+\alpha+0 \in T_x E^* \mid \alpha \in (\mathfrak{k}_h/\mathfrak{k}_q)\} \subset T_x\mathfrak{X}$, which completes the proof of the theorem. □

We make some comments on the somewhat peculiar condition $K_p \subset K_f$. In this way stated, this condition depends on a point (f, p) of the coadjoint orbit, but as one easily verifies, it implies that

$$\kappa \cdot h - h \in \operatorname{im} \tau_q^* \quad \forall \kappa \in K_q \tag{5.1}$$

for an arbitrary point $(h, q) \in \mathcal{O}_\nu^G$. Now, condition (5.1) is equivalent to saying that all the little-group orbits are trivial because $i_q^*(\kappa \cdot h) = i_q^*(h)\ \forall \kappa \in K_q$. Otherwise stated, condition $K_p \subset K_f$ implies that the orbit Z coincides with the bundle of little-group orbits Y.

The converse now is not in general true, that is the condition

$$Z = Y$$

does not in general implies the existence of an element $(f, p) \in \mathcal{O}_\nu^G$ such that $K_p \subset K_f$. Let us explain why. If $Z = Y$ then condition (5.1) is valid, so given $(h, q) \in \mathcal{O}_\nu^G$ and $\kappa \in K_q$, we have $\kappa \cdot h - h = q \odot v(\kappa)$, where $v : K_q \to V$ is such that its equivalence class $[v] : K_q \to V/\ker \tau_q^*$ belongs to $Z^1(K_q, V/\ker \tau_q^*)$. This is evident by direct calculation using the fact that $\ker \tau_q^*$ is K_q-invariant, which induces a representation $K_q \to GL(V/\ker \tau_q^*)$. If now there exists an element $(h, q) \in \mathcal{O}_\nu^G$ such that $\mathrm{H}^1(K_q, V/\ker \tau_q^*) = 0$, then we can find an element (f, p) of the same orbit with the property $K_p \subset K_f$. In fact, in this case we have always $q \odot v(\kappa) = q \odot (\kappa \cdot v_0 - v_0)$ for a fixed element $v_0 \in V$. Choosing thus $p = q$ and $f = h - q \odot v_0$, we have the desired result.

6. Pukanszky's condition and the semidirect product

Let us first state some definitions and results [3] about polarizations and Pukanszky's condition that will be used in the sequel.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and ν an element of $\mathfrak{g}^*$. Given a subspace $\mathfrak{a} \subset \mathfrak{g}$ which contains the Lie algebra $\mathfrak{g}_\nu$ of the isotropy subgroup G_ν with respect to the coadjoint action, we define the symplectic orthogonal $\mathfrak{a}^\perp$ by

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g} \mid \nu([X, Y]) = 0,\ \forall Y \in \mathfrak{a}\}. \tag{6.1}$$

If we note by $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$ and by $\mathfrak{g}^c \ni \mu \mapsto \bar{\mu} \in \mathfrak{g}^c$ the complex conjugation, we may extend this notion immediately for subspaces of $\mathfrak{g}^c$ which contain $\mathfrak{g}_\nu^c$.

We say now that the complex Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}^c$ is a polarization with respect to $\nu \in \mathfrak{g}^*$ if $\mathfrak{h}$ contains $\mathfrak{g}_\nu^c$, is invariant under the adjoint action of G_ν, $\mathfrak{h}^\perp = \mathfrak{h}$ and $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}^c$. Each algebraic polarization $\mathfrak{h}$ corresponds to a G-invariant geometric polarization $\mathcal{F}$, the correspondence being given by $\mathfrak{h} \cdot \nu = \mathcal{F}_\nu \subset (T_\nu \mathcal{O}_\nu^G)^c$. The condition on the symplectic orthogonal $\mathfrak{h}^\perp$ can be restated as follows:

$$(\mathfrak{h}^\perp = \mathfrak{h}) \Longleftrightarrow (\dim_{\mathbb{C}} \mathfrak{h} = \tfrac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}_\nu) \quad \text{and} \quad \nu([\mathfrak{h}, \mathfrak{h}]) = 0). \tag{6.2}$$

To each polarization $\mathfrak{h}$ we can associate two real Lie subalgebras $\mathfrak{d} \subset \mathfrak{e}$ of $\mathfrak{g}$ defined by

$$\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}. \tag{6.3}$$

We denote also by $D_0 \subset E_0$ the connected Lie subgroups of G whose Lie algebras are $\mathfrak{d}$ and $\mathfrak{e}$, respectively. The conditions on the polarization $\mathfrak{h}$ ensure that the subsets $D = D_0 \cdot G_\nu \subset E = E_0 \cdot G_\nu$ are subgroups of G.

Lemma 6.1 [7]. (1) $\mathfrak{d}^\perp = \mathfrak{e}$; (2) *the groups D and D_0 are closed Lie subgroups of G with the same Lie algebra $\mathfrak{d}$;* (3) *$i_{\mathfrak{d}}^* \nu$ is invariant under the coadjoint action of D;* (4) *$\mathfrak{e}^\circ$ and $\nu + \mathfrak{e}^\circ$ are invariant under the coadjoint action of the subgroup D;* (5) *if E is a Lie subgroup of G, then its Lie algebra is $\mathfrak{e}$.*

Pukanszky's condition now, is a supplementary condition on the polarization $\mathfrak{h}$. The following lemma gives three equivalent variants of this condition.

Lemma 6.2 (Pukanszky's condition, [3]). *The following conditions are equivalent:* (1) $\nu + \mathfrak{e}^\circ \subset \mathcal{O}_\nu^G$; (2)$D \cdot \nu = \nu + \mathfrak{e}^\circ$; (3)$D \cdot \nu$ *is closed in* $\mathfrak{g}^*$.

Consider now the case where the Lie group G is a semidirect product, $G = K \times_\rho V$ (notation of Section 2). Then, its Lie algebra is $\mathfrak{g} = \mathfrak{k} \oplus_\rho V$ and the corresponding complexified Lie algebra $\mathfrak{g}^c = \mathfrak{k}^c \oplus_\rho V^c$. We are interested in polarizations of $\mathfrak{g}^c$ (with respect to $\nu \in \mathfrak{g}^*$) which are of the form $\mathfrak{h} = \mathfrak{a} \oplus_\rho V^c$, $\mathfrak{a} \subset \mathfrak{k}^c$. Although this type of polarization seems to be very special, it leads to quite interesting results as we shall see in the sequel.

We examine first the restrictions imposed to the subspace $\mathfrak{a}$ by the fact that $\mathfrak{h}$ is a polarization. We find successively:

(1) *$\mathfrak{h}$ is a subalgebra of* $\mathfrak{g}^c$. Then, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ which implies $[\mathfrak{a}, \mathfrak{a}] \oplus_\rho [\mathfrak{a}, V^c] \subset \mathfrak{a} \oplus_\rho V^c$. Thus, $\mathfrak{a}$ must be a Lie subalgebra of $\mathfrak{k}^c$.

(2) $\mathfrak{h}^\perp = \mathfrak{h}$. Equivalently, we have relation (6.2). By direct calculation of the dimensions appearing in (6.2), we obtain: $\dim_{\mathbb{C}} \mathfrak{h} = \dim_{\mathbb{C}} \mathfrak{a} + \dim V$ and $\dim \mathfrak{g} + \dim \mathfrak{g}_\nu = 2 \dim V + \dim \mathfrak{k}_p + \dim (\mathfrak{k}_p)_\phi$. Thus,

$$\dim_{\mathbb{C}} \mathfrak{a} = \tfrac{1}{2}(\dim \mathfrak{k}_p + \dim (\mathfrak{k}_p)_\phi). \tag{6.4}$$

We have one more restriction coming from the condition $\nu([\mathfrak{h}, \mathfrak{h}]) = 0$. Indeed, using relations (4.2) and (4.3), this condition gives: $A \cdot p = 0$ and $A \cdot f + p \odot a \in \mathfrak{a}^\circ$, $\forall A \in \mathfrak{a}$, $a \in V^c$. But if this is the case, we have $\mathfrak{a} \subset \mathfrak{k}_p^c$ and since $p \odot a \in (\operatorname{im} \tau_p^*)^c = (\mathfrak{k}_p^c)^\circ$ (Lemma 2.1), for each $a \in V^c$, we obtain necessarily $A \cdot f \in \mathfrak{a}^\circ$, $\forall A \in \mathfrak{a}$, or, $f([\mathfrak{a}, \mathfrak{a}]) = 0$. Taking into account relation (6.4) as well as the facts $\mathfrak{a} \subset \mathfrak{k}_p^c$ and $f([\mathfrak{a}, \mathfrak{a}]) = 0$, we conclude that $\mathfrak{a}^\perp = \mathfrak{a}$ (the symplectic orthogonal being taken with respect to $\phi = i_p^* f$).

(3) $\mathfrak{h}$ *is invariant under the adjoint action of* G_ν. By Eq. (2.4) it is immediate that $\mathfrak{a}$ must be invariant under the adjoint action of $(K_p)_\phi$.
(4) $\mathfrak{h} + \bar{\mathfrak{h}}$ *is a Lie subalgebra of* $\mathfrak{g}^c$. Equivalently, it suffices to demand $[\mathfrak{h}, \bar{\mathfrak{h}}] \subset \mathfrak{h} + \bar{\mathfrak{h}}$, because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}^c$. But the last bracket is equal to $[\mathfrak{a}, \bar{\mathfrak{a}}] \oplus [\mathfrak{a}+\bar{\mathfrak{a}}, V^c]$; thus we require only $[\mathfrak{a}, \bar{\mathfrak{a}}] \subset \mathfrak{a} + \bar{\mathfrak{a}}$, because the last two terms belong already to V^c.

Conversely, suppose that $\mathfrak{a}$ is a polarization of $\mathfrak{k}_p^c$ (with respect to $\phi = i_p^* f$). In that case, it is easy to reverse the previous reasonings and deduce that $\mathfrak{h} = \mathfrak{a}\oplus_\rho V^c$ is a polarization (with respect to ν) for the Lie algebra $\mathfrak{g}^c$ (see also [11]).

Proposition 6.3. *Let* $G = K \times_\rho V$ *be a semidirect product,* $\nu = (f, p) \in \mathfrak{g}^* = \mathfrak{k}^*\oplus_\rho V^*$ *an element of the dual of its Lie algebra and* $\mathfrak{g}_\nu$ *the isotropy subalgebra of* ν *with respect to the coadjoint action. Then a subspace* $\mathfrak{h} = \mathfrak{a}\oplus_\rho V^c \subset \mathfrak{g}^c$ *is a polarization for the Lie algebra* $\mathfrak{g}^c$ *with respect to* ν *if and only if* $\mathfrak{a}$ *is a polarization of* $\mathfrak{k}_p^c$ *with respect to* $\phi = i_p^* f$.

Next, we show that an analogous phenomenon occurs with respect to the validity of Pukanszky's condition. More precisely, the validity of Pukanszky's condition for polarizations of a semidirect product $G = K \times_\rho V$ described in Proposition 6.3, reduces to validity of the same condition for polarizations of the isotropy Lie subalgebra $\mathfrak{k}_p$.

Suppose that the polarization $\mathfrak{h} = \mathfrak{a}\oplus_\rho V^c$ of $\mathfrak{g}^c$ satisfies Pukanszky's condition. For the case of the semidirect product we are interested in, the (real) subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ (see Eq. (6.3)) will be: $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g} = \mathfrak{a} \cap \mathfrak{k}\oplus_\rho V = \mathfrak{p}\oplus_\rho V$ and $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g} = (\mathfrak{a} + \bar{\mathfrak{a}}) \cap \mathfrak{k} \oplus_\rho V = \mathfrak{q}\oplus_\rho V$. The connected Lie subgroup D_0 whose Lie algebra is equal to $\mathfrak{d}$ is a semidirect product $D_0 = P_0 \times_\rho V$, where P_0 is the closed, connected and simply connected subgroup of K_p whose Lie algebra is $\mathfrak{p}$. The validity of Pukanszky's condition for the polarization $\mathfrak{h}$ implies the relation $D_0 \cdot \nu = \nu + \mathfrak{e}^\circ$ (see Lemma 6.2(2)). Consider then the element $d = (\lambda, v) \in D_0$ and let $\nu = (f, p)$ as previously: $d\cdot\nu = (\lambda\cdot f + \lambda\cdot p \odot v, \lambda\cdot p)$. We know that the difference $d \cdot \nu - \nu$ must be contained in $\mathfrak{e}^\circ$, so

$$(\lambda \cdot f - f + \lambda \cdot p \odot v) \in \mathfrak{q}^\circ \tag{6.5}$$

and

$$(\lambda \cdot p - p) \in V^\circ = 0. \tag{6.6}$$

It follows that $\lambda \cdot p = p \Rightarrow \lambda \in K_p$ which is already satisfied because $\mathfrak{k}_p \supset \mathfrak{p} = \mathfrak{a} \cap \mathfrak{k} \supset (\mathfrak{k}_p)_\phi$ and P_0 is connected. Furthermore $(\lambda \cdot f - f + p \odot v) \in \mathfrak{q}^\circ$; but $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{k}_p \Rightarrow \mathfrak{p}^\circ \supset \mathfrak{q}^\circ \supset \mathfrak{k}_p^\circ$ and we obtain $p \odot v \in \operatorname{im} \tau_p^* = \mathfrak{k}_p^\circ \subset \mathfrak{q}^\circ$. Thus, we must have $(\lambda \cdot f - f) \in \mathfrak{q}^\circ$, or $(\lambda \cdot \phi - \phi) \in i_p^*\mathfrak{q}^\circ$.

As a result, the condition $D_0 \cdot \nu = \nu + \mathfrak{e}^\circ$ (Pukanszky's condition) implies $P_0 \cdot \phi = \phi + i_p^*\mathfrak{q}^\circ$ which is exactly Pukanszky's condition for the polarization $\mathfrak{a}$ of $\mathfrak{k}_p^c$ because $i_p^*\mathfrak{q}^\circ$ is the annihilator of $\mathfrak{q}$ in $\mathfrak{k}_p$. Finally, the previous analysis shows easily that the converse is also true, that is if the polarization $\mathfrak{a}$ of $\mathfrak{k}_p^c$ satisfies Pukanszky's condition, then the same is true for the polarization $\mathfrak{h} = \mathfrak{a}\oplus_\rho V^c$ of $\mathfrak{g}^c$. We have thus proved:

Theorem 6.4. *Let $\mathfrak{h} = \mathfrak{a} \oplus_\rho V^c$ be a polarization of the complexified Lie algebra $\mathfrak{g}^c$ of a semidirect product $G = K \times_\rho V$ with respect to an element $\nu = (f, p) \in \mathfrak{g}^*$; equivalently $\mathfrak{a}$ is a polarization of $\mathfrak{k}_p^c$ with respect to $\phi = i_p^* f$. Then, $\mathfrak{h}$ satisfies Pukanszky's condition if and only if $\mathfrak{a}$ satisfies it as well.*

Scholium 6.5. The Lie algebra $\mathfrak{g}$ of a semidirect product is a special case of extension of a Lie algebra $\mathfrak{k}$ by an abelian Lie algebra V. The complex Lie subalgebra $\mathfrak{h}$ studied in this section is also of this special type. But one can reconsider Proposition 6.3 and Theorem 6.4 in the following way. Let $\mathfrak{g}^c = \mathfrak{k}^c \oplus_\rho V^c$ and $\mathfrak{h} \subset \mathfrak{g}^c$ be a Lie subalgebra. If $\mathfrak{a}$ is the image of the homomorphism $\mathfrak{g}^c \to \mathfrak{k}^c$ restricted to $\mathfrak{h}$, then $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{k}^c$, see (2.2). In other words, $\mathfrak{h}$ is an extension of $\mathfrak{a}$ by a vector subspace of V^c, the kernel of $\mathfrak{h} \to \mathfrak{a}$. Considering now Lie subalgebras $\mathfrak{h}$ which are extensions of Lie subalgebras of $\mathfrak{k}^c$ by V^c,

$$0 \longrightarrow V^c \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{a} \longrightarrow 0, \tag{6.7}$$

one easily verifies that Proposition 6.3 and Theorem 6.4 remain still valid if we replace simply $\mathfrak{h} = \mathfrak{a} \oplus_\rho V^c$ by the exact sequence (6.7). Finally, we note that the corresponding geometric polarization belongs to the category of polarizations studied in [11].

As an illustration, we then construct explicitly, for an arbitrary semidirect product, a polarization "trivial" in some sense, which satisfies Pukanszky's condition (making of course appropriate choices). Thanks to Proposition 6.3 and Theorem 6.4, it is sufficient to construct a polarization $\mathfrak{a}$ of $\mathfrak{k}_p$ satisfying the same condition.

Suppose that $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$ and let $\mathfrak{a} = \bar{\mathfrak{a}} = \mathfrak{k}_p^c$. Clearly, $\mathfrak{a}$ is a real subalgebra of $\mathfrak{k}_p^c$, invariant under the adjoint action of the isotropy subgroup $(K_p)_\phi$ which, in this case, is the union of connected components of K_p. Furthermore, $2 \dim_{\mathbb{C}} \mathfrak{a} = 2 \dim \mathfrak{k}_p = \dim \mathfrak{k}_p + \dim (\mathfrak{k}_p)_\phi$, for $(\mathfrak{k}_p)_\phi = \{A \in \mathfrak{k}_p \mid A \cdot \phi = 0\} = \mathfrak{k}_p$ and the Lie subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ coincide: $\mathfrak{p} = \mathfrak{q} = \mathfrak{k}_p$. Finally, the element $\phi = i_p^* f$ vanishes on all the brackets $[A, B]$, for each $A, B \in \mathfrak{k}_p$, because we have always $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$. We deduce that $\mathfrak{a} = \mathfrak{k}_p^c$ is a (real) polarization of $\mathfrak{k}_p$.

Next, for each $\lambda \in P_0 = (K_p)_0$ we find: $\lambda \cdot \phi = \phi$, so $\lambda \cdot \phi = \phi + i_p^* \mathfrak{q}^\circ$ (here $i_p^* \mathfrak{q}^\circ = i_p^* \mathfrak{k}_p^\circ = 0$ is the annihilator of $\mathfrak{k}_p$ in $\mathfrak{k}_p^*$).

Corollary 6.6. *For each semidirect product $G = K \times_\rho V$, the coadjoint orbit of the element $\nu = (f, p) \in \mathfrak{g}^*$ with $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$ admits a real polarization satisfying Pukanszky's condition.*

7. Symplectic induction

We recall here the method of symplectic induction [5] which will be very important to a deeper geometrical investigation of the coadjoint orbits of semidirect products. More on this method can be found in [3]. Notice that the induction of Hamiltonian actions appears independently in [16].

Let (M, ω) be a symplectic manifold and H a closed Lie subgroup of a Lie group G. Suppose we have a (left) Hamiltonian action $\Phi : H \times M \to M$ which admits an equivariant momentum map $J_M : M \to \mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of H. The aim of the symplectic induction method is to construct a symplectic manifold, denoted as M_{ind}, on which the group G acts in a Hamiltonian way with equivariant momentum map $J_{\text{ind}} : M_{\text{ind}} \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G.

In order to construct the Hamiltonian space $(M_{\text{ind}}, \omega_{\text{ind}}, G, J_{\text{ind}})$, we proceed as follows. Using the natural isomorphism $T^*G \cong G \times \mathfrak{g}^*$ (obtained by identifying $\mathfrak{g}^*$ with the left-invariant 1-forms on G), we obtain a left action $\check{\Phi}$ of H on $\check{M} = M \times T^*G$ given by

$$\check{\Phi}_h(m, g, \mu) = (\Phi_h(m), gh^{-1}, h \cdot \mu), \quad \forall h \in H, \quad (m, g, \mu) \in M \times T^*G. \tag{7.1}$$

This action is symplectic for the symplectic structure $\check{\omega} = \pi_1^*\omega_M + \pi_2^* \,\mathrm{d}\theta_G$ on $\check{M}$ if $\pi_1 : \check{M} \to M$ and $\pi_2 : \check{M} \to T^*G$ are the projections: $\check{\Phi}_h^*\check{\omega} = \check{\omega}$; $\check{\Phi}$ is also proper because H is closed. Furthermore, it admits an equivariant momentum map $J_{\check{M}} : \check{M} \to \mathfrak{h}^*$ equal to $J_{\check{M}} = \pi_1^* J_M + \pi_2^* J_H$, where J_H is the momentum map for the cotangent lift of the right action of H on G. If $i_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the inclusion and $i_{\mathfrak{h}}^* : \mathfrak{g}^* \to \mathfrak{h}^*$ the corresponding projection, then this momentum map is given by

$$J_H(g, \mu) = -i_{\mathfrak{h}}^* \mu.$$

The element $0 \in \mathfrak{h}^*$ is a regular value for the momentum map $J_{\check{M}}$ and so the quotient $M_{\text{ind}} = J_{\check{M}}^{-1}(0)/H$ will be a symplectic manifold (Marsden–Weinstein reduction). We call M_{ind} induced symplectic manifold and we denote it by $\mathrm{Ind}_H^G M$; ω_{ind} will denote the symplectic structure of M_{ind}.

In order to obtain a Hamiltonian action of G on M_{ind} we let the group G act trivially on M; we consider also the canonical lift to T^*G of the left multiplication on G. Then we have a Hamiltonian action of G on $\check{M}$ with equivariant momentum map $\check{J} : \check{M} \to \mathfrak{g}^*$ given by

$$\check{J}(m, g, \mu) = g \cdot \mu. \tag{7.2}$$

This action commutes with the action of H on $\check{M}$ and leaves invariant the momentum map $J_{\check{M}}$, so a symplectic action of G is induced on M_{ind}. Since the momentum map $\check{J}$ is H-invariant, it descends as an equivariant momentum map $J_{\text{ind}} : M_{\text{ind}} \to \mathfrak{g}^*$ for the action of G on $M_{\text{ind}} = \mathrm{Ind}_H^G M$.

Proposition 7.1 [3]. *The induced symplectic manifold $M_{\text{ind}} = \mathrm{Ind}_H^G M$ is a fibre bundle over $T^*(G/H)$ with typical fibre the symplectic manifold M. Moreover, the restriction of ω_{ind} to a fibre yields the original symplectic structure ω_M on M.*

Let us note here that the symplectic induction procedure can be carried out without using the trivialization of T^*G, see [8].

If we perform now the symplectic induction for $M =$ point, then the induced symplectic manifold is isomorphic as a manifold to $T^*(G/H)$; we can extend the isomorphism to the symplectic category if we modify the natural symplectic structure $\mathrm{d}\theta_{G/H}$ of $T^*(G/H)$ by a "magnetic" term, that is by the pull-back of an appropriate closed 2-form $\beta_0 \in \Omega^2(G/H)$, [3]. Thus, the symplectic induction over a point leads to the modified cotangent bundle $(T^*(G/H), \mathrm{d}\theta_{G/H} + \tau^*\beta_0)$, where $\tau : T^*(G/H) \to G/H$ is the cotangent projection.

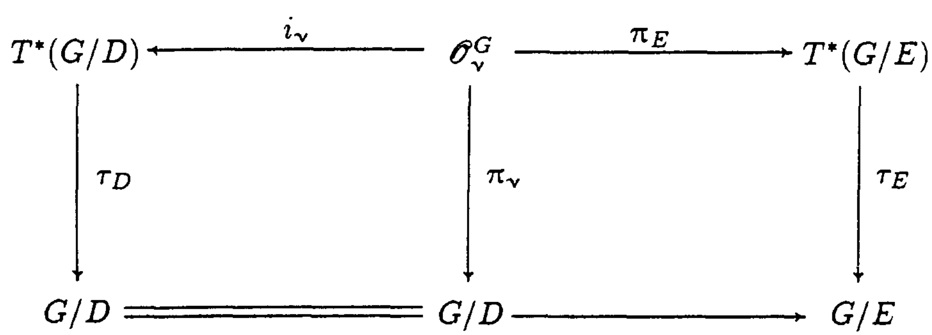
8. The structure of coadjoint orbits

We will use now the results of [3] on the structure of the coadjoint orbits endowed with a polarization satisfying Pukanszky's condition in order to analyse further the structure of the semidirect product coadjoint orbits.

Let us recall first the principal results of this article. If G is a Lie group, $\nu \in \mathfrak{g}^*$ an element of the dual of its Lie algebra and $\mathfrak{h}$ a polarization with respect to ν, we have two real Lie subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ of $\mathfrak{g}$ canonically associated to $\mathfrak{h}$ (see relations (6.3)). If we define $\mathcal{F} = (G \times (\nu + \mathfrak{e}^\circ))/D$ (for the property of the group D see Lemma 6.1), then $\mathcal{F}$ is a vector subbundle of $T^\sharp(G/D)$ with symplectic form $\omega_{\mathcal{F}}$ obtained by restriction of the symplectic form of $T^\sharp(G/D)$; further, there is a symplectic action of G on $\mathcal{F}$ admitting a momentum map $J_{\mathcal{F}} : \mathcal{F} \to \mathfrak{g}^*$ which may be calculated via (7.2).

Proposition 8.1. *The following four conditions on the polarization are equivalent:*
(1) *Pukanszky's condition.*
(2) *The momentum map $J_{\mathcal{F}} : \mathcal{F} \to \mathfrak{g}^*$ is onto $\mathcal{O}_\nu^G$.*
(3) *The symplectic action of G on $\mathcal{F}$ is transitive.*
(4) *$J_{\mathcal{F}}$ is a symplectomorphism between $(\mathcal{F}, \omega_{\mathcal{F}})$ and $\mathcal{O}_\nu^G$.*

Proposition 8.2. *If E is a closed subgroup of G and if Pukanszky's condition is satisfied, then there exists a commutative diagram*



with the following properties:
(1) *(i_ν, π_ν) is the identification of the coadjoint orbit $\mathcal{O}_\nu^G$ as a symplectic subbundle of $T^*(G/D)$ according to Proposition* 8.1.
(2) *$\pi_E : \mathcal{O}_\nu^G \to T^*(G/E)$ is a fibre bundle whose fibres together with the restricted symplectic form, are symplectomorphic to the (pseudo-)Kähler space E/D.*

Consider now the case where the Lie group G is a semidirect product, $G = K \times_\rho V$ and let $\mathfrak{h}$ be a polarization of $\mathfrak{g}^*$ with respect to $\nu = (f, p) \in \mathfrak{g}^*$. We are always interested in polarizations of the form $\mathfrak{h} = \mathfrak{a} \oplus_\rho V^c$ satisfying Pukanszky's condition. By Proposition 6.3, $\mathfrak{a}$ is a polarization of $\mathfrak{k}_p^c$ with respect to $\phi = i_p^* f$. Furthermore, Pukanszky's condition is equivalently satisfied by $\mathfrak{a}$. Applying Proposition 8.1, the following result is immediate:

Corollary 8.3. *A necessary and sufficient condition for the symplectic subbundle $\mathcal{F} = (G \times (\nu + \mathfrak{e}^\circ))/D \subset T^\sharp(G/D)$ to be symplectomorphic to the coadjoint orbit $\mathcal{O}_\nu^G$, is that the symplectic subbundle $\mathcal{F}_p = (K_p \times (\phi + i_p^* \mathfrak{q}^\circ))/P \subset T^\sharp(K_p/P)$ be symplectomorphic to the little-group coadjoint orbit $\mathcal{O}_\phi^{K_p}$.*

We determine now an equivalence class in the quotient $\mathcal{F} = (G \times (\nu + \mathfrak{e}^\circ))/D$, given that $\mathfrak{e}^\circ = (\mathfrak{q} \oplus_\rho V)^\circ \cong \mathfrak{q}^\circ \subset \mathfrak{k}^* \times \{0\}$ and that D is described by the exact sequence

$$0 \longrightarrow V \xrightarrow{\ i\ } D \xrightarrow{\ j\ } P \longrightarrow e. \tag{8.1}$$

Let $(g, \nu + w) \in G \times (\nu + \mathfrak{e}^\circ)$ and $[(g, \nu + w)]$ be its equivalence class. We know that $\mathcal{F}$ is an affine bundle associated to the principal fibre bundle $G \to G/D$. So, it is sufficient to find the equivalence class $[g]$. But thanks to the exact sequence (8.1), we obtain $[g] = g \cdot D = [\kappa] = \kappa \cdot P$ if $g = (\kappa, v) \in G$ and we may write $\mathcal{F} = (K \times (f + \mathfrak{q}^\circ))/P$. Furthermore, there is a canonical inclusion $K_p/P \hookrightarrow K/P$ induced by the inclusion of the closed subgroup K_p in K; we have so a projection $T_x^*(K/P) \to T_x^*(K_p/P)$, for $x = [\kappa]$, $\kappa \in K_p$.

Let us examine in more detail this projection of cotangent spaces. If we denote $T_e L_\kappa(\mathfrak{p}) \subset T_\kappa K$ as $\mathfrak{p}_\kappa$ (recall that $\mathfrak{p} = \mathfrak{a} \cap \mathfrak{k}$), then clearly $T_x(K/P) \cong T_\kappa K/\mathfrak{p}_\kappa$ and $T_x^*(K/P) \cong \mathfrak{p}_\kappa^\circ \subset \mathfrak{k}_\kappa^*$, the annihilator of $\mathfrak{p}_\kappa$. Similarly, $T_x(K_p/P) \cong T_\kappa K_p/\mathfrak{p}_\kappa$ and $T_x^*(K_p/P)$ is isomorphic to the space of elements of T^*K_p which vanish on $\mathfrak{p}_\kappa$. If $T_\kappa i_p : T_\kappa K_p \hookrightarrow T_\kappa K$ is the natural inclusion, we may write $T_\kappa^*(K_p/P) \cong (T_\kappa i_p)^* \mathfrak{p}_\kappa^\circ$. Now, the inclusion $\mathfrak{p} \subset \mathfrak{q}$ implies $\mathfrak{q}_\kappa^\circ \subset \mathfrak{p}_\kappa^\circ$; but $\mathfrak{q}_\kappa^\circ$ and $(T_\kappa i_p)^* \mathfrak{q}_\kappa^\circ$ are the fibres of $\mathcal{F} \cong \mathcal{O}_\nu^G$ and $\mathcal{F}_p \cong \mathcal{O}_\phi^{K_p}$, respectively, over x (see also Corollary 8.3). Thus, the fibres of $\mathcal{F}_p$ are obtained from those of $\mathcal{F}$ under the projections $(T_\kappa i_p)^*$. Therefore:

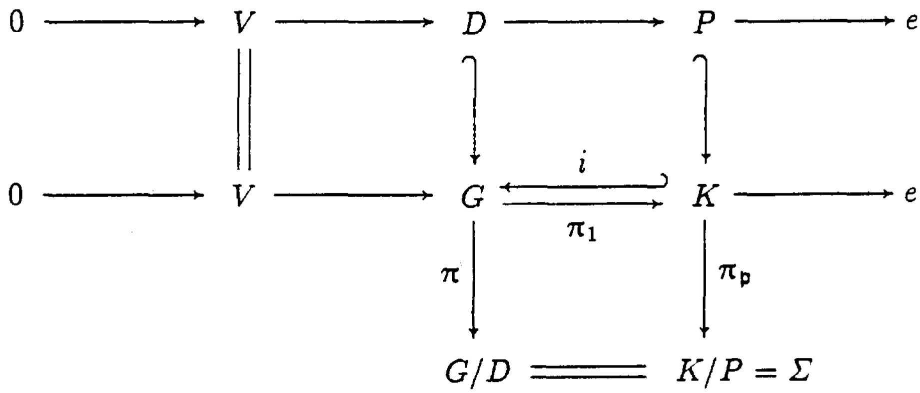
Corollary 8.4. *Validity of Pukanszky's condition for a polarization $\mathfrak{h} = \mathfrak{a} \oplus_\rho V^c$ at $\nu \in \mathfrak{g}^*$ implies that the coadjoint orbit $\mathcal{O}_\nu^G$ is symplectomorphic to the quotient $\mathcal{F} = (K \times (f + \mathfrak{q}^\circ))/P$ and that the coadjoint orbit $\mathcal{O}_\phi^{K_p}$ is obtained by restricting $\mathcal{F} \cong \mathcal{O}_\nu^G$ to the closed subset $K_p/P \subset K/P$ and projecting its fibres by the natural projection between the corresponding cotangent bundles.*

Clearly, under the conditions of Proposition 8.2, the coadjoint orbit $\mathcal{O}_\phi^{K_p}$ has properties analogous to those described in this proposition because, according to Theorem 6.4, Pukanszky's condition on $\mathfrak{h}$ is equivalent to the same condition on the polarization $\mathfrak{a}$.

9. Connections and symplectic induction by semidirect products

We consider in this section the problem of the arbitrary choice of the connection in the symplectic induction process, pointed out in [3]. This connection plays the role of a Yang–Mills potential in the more general geometrical interaction scheme due to Guillemin–Sternberg [4,14] and Weinstein [15]. The symplectic induction is a special case of this model and the arbitrariness or not of the connection has been related to the localization of relativistic particles in [2,3].

We restrict our attention to the case where we have a principal fibre bundle $\pi : G \to G/D$, formed by a semidirect product $G = K \times_\rho V$ and a closed subgroup D belonging to the set of extensions of the Lie subgroup $P \subset K$ by the vector group V. In this case, the base space G/D is equal to K/P; we denote this quotient by Σ. The following commutative diagram illustrates this situation.



Let then $\gamma \in \Omega^1(G) \otimes \mathfrak{d}$ be a connection on $\pi : G \to G/D$, where $\mathfrak{d}$, the Lie algebra of D, is an extension of the Lie algebra $\mathfrak{p}$ by V. Set now $g = (\kappa, v) \in G$, $h = (\lambda, u) \in D$ and $\xi = (B, b) \in \mathfrak{d}$. Then, if $R_h(g) = gh$ is the right action of D on G, by the defining properties of a connection form we must have $R_h^*\gamma = \mathrm{Ad}(h^{-1}) \circ \gamma$ and $\gamma_g(\widehat{\xi}(g)) = \xi$, where $\widehat{\xi}$ is the fundamental vector field of the Lie algebra element ξ for the right action on G: $\widehat{\xi}(g) = (\widehat{B}(\kappa), \kappa \cdot b)$. Using the fact that the $\mathfrak{p}$-components of TR_h and $\mathrm{Ad}(h^{-1})$ are TR_λ and $\mathrm{Ad}(\lambda^{-1})$, respectively, we obtain immediately that the $\mathfrak{p}$-component $\alpha = (i^* \otimes T_e\pi_1) \circ \gamma$ of the pull-back of γ under the inclusion $i : K \hookrightarrow G$ is a connection 1-form on $K \to K/P$.

Conversely, suppose that we have a connection $\alpha \in \Omega^1(K) \otimes \mathfrak{p}$ on $K \to K/P$, then, for each $\kappa \in K$, we have a horizontal subspace $H_\kappa \subset T_\kappa K$ setting $H_\kappa = \ker \alpha_\kappa$; H_κ is isomorphic to $T_q\Sigma$ under the projection $\pi_\mathfrak{p} : K \to \Sigma$, $q = \pi_\mathfrak{p}(\kappa) = [\kappa]$. If now $g = (\kappa, v) \in G$, one can define a subspace $\bar{H}_g \subset T_g G \cong T_\kappa K \oplus V$, complementary to the vertical subspace at g and isomorphic also to $T_q\Sigma$, $q = \pi(g) = [g] = [\kappa]$. In fact, if $X \in H_\kappa$, let us define $\bar{X} \in \bar{H}_g$ by

$$\bar{X} = (X, T_e\rho(T_\kappa R_{\kappa^{-1}}(X))v). \tag{9.1}$$

We set for simplicity $T_e\rho(T_\kappa R_\kappa^{-1}(X))v = R_{\kappa^{-1}}X \cdot v$. In order to prove that $\bar{H}$ defines a connection on $\pi : G \to \Sigma$, it is sufficient to check if $\bar{H}$ is invariant under the right action of D on G. Taking $h = (\lambda, u) \in D$, we find: $TR_h(\bar{X}) = (\widetilde{X}, w)$, where $\widetilde{X} = TR_\lambda(X)$ and $w = T_\kappa\rho(X)u + R_{\kappa^{-1}}X \cdot v$. But easy calculation shows that $w = R_{(\kappa\lambda)^{-1}}\widetilde{X} \cdot (\kappa \cdot u + v)$; consequently, $TR_h(\bar{X}) \in \bar{H}_{gh}$, which proves that $\bar{H}$ is indeed a connection.

In order to calculate now the corresponding connection form γ, we use the decomposition $(Z, w) = (X + \widehat{B}(\kappa), R_{\kappa^{-1}}X \cdot v + \kappa \cdot b)$ of an arbitrary tangent vector at (κ, v) into horizontal and vertical parts, $(B, b) \in \mathfrak{d}$, $X \in \ker\alpha_\kappa$. Then γ must satisfy $\gamma(Z, w) = (B, b)$ and if we decompose γ as $\gamma = (\gamma_1, \Delta)$, then $\gamma_1 = \pi_1^*\alpha$ and $\Delta \in \Omega^1(G) \otimes V$ is given by

$$\Delta_{(\kappa,v)}(Z, w) = \kappa^{-1} \cdot (w - R_{\kappa^{-1}}X \cdot v). \tag{9.2}$$

We have thus proved:

Proposition 9.1. *If $\gamma \in \Omega^1(G) \otimes \mathfrak{d}$ is a connection form on the principal bundle $G \to G/D$, then $\alpha = (i^* \otimes T_e\pi_1) \circ \gamma$, the $\mathfrak{p}$-component of the pull-back of γ under the inclusion $i : K \hookrightarrow G$, is a connection form on $\pi : K \to \Sigma$. Furthermore, a connection $\alpha \in \Omega^1(K) \otimes \mathfrak{p}$ determines a preferred connection $\gamma \in \Omega^1(G) \otimes \mathfrak{d}$. Indeed, γ is equal to $(\pi_1^*\alpha, \Delta)$, where Δ is given by Eq.* (9.2) *and the horizontal spaces defined by γ are given by Eq.* (9.1).

Remark 9.2. Suppose that G and D are such that we can apply symplectic induction from a point $\nu_0 \in \mathfrak{d}^*$. Then we know [3] that $\gamma_0 = \nu_0 \circ \gamma$ is a 1-form invariant on the fibres of π and therefore, there exists a 2-form $\beta_0 \in \Omega^2(\Sigma)$ such that $\pi^*\beta_0 = \mathrm{d}\gamma_0$. This 2-form gives the modification term of the canonical symplectic structure of the cotangent bundle $T^*(G/D) = T^*\Sigma$. Now if there exists a canonical connection $\alpha \in \Omega^1(K) \otimes \mathfrak{p}$, then we may choose γ in a natural way according to Proposition 9.1 and so β_0 is also canonical.

We examine now a special case where the choice of the connection γ is guided by supplementary geometrical structures (so we have a canonical connection). The following proposition explains then a result of [2] concerning the localization procedure of relativistic particles with non-zero mass. This is precisely the case of massive Poincaré coadjoint orbits (the hyperboloid $m =$ const. > 0 is a symmetric space unlike the massless case where the light cone is not).

Proposition 9.3. *Let $G = K \times_\rho V$ be a semidirect product, $\mathfrak{g} = \mathfrak{k} \oplus_\rho V$ its Lie algebra and $\nu = (f, p) \in \mathfrak{g}^*$. Suppose there exists a polarization $\mathfrak{a}$ of the complexified Lie algebra $\mathfrak{k}_p^c$ satisfying Pukanszky's condition and K_p/P and $Z = K/K_p$ are symmetric spaces, where P is the closed subgroup of K_p determined by $\mathfrak{a}$ (Lemma* 6.1). *Then the coadjoint orbit $\mathcal{O}_\nu^G$ is symplectomorphic to a symplectic subbundle of a modified cotangent bundle (Proposition* 8.1) *whose symplectic structure is canonical.*

Proof. The only thing we have to prove is that the modified symplectic structure of the cotangent bundle $T^\natural(G/D)$ is canonical or, equivalently, that the connection form α is canonical.

Since K_p/P and $Z = K/K_p$ are symmetric spaces, there exist involutive automorphisms $I_p : K_p \to K_p$ and $I : K \to K$ defining the canonical symmetric space decompositions $\mathfrak{k}_p = \mathfrak{p} \oplus \mathfrak{m}$ and $\mathfrak{k} = \mathfrak{k}_p \oplus \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the subspaces of $\mathfrak{k}_p$ and $\mathfrak{k}$, respectively, corresponding to the eigenvalue -1 of $T_e I_p$ and $T_e I$.

Using the known property $\mathrm{Ad}(P)\mathfrak{m} \subset \mathfrak{m}$ and $\mathrm{Ad}(K_p)\mathfrak{n} \subset \mathfrak{n}$ of these subspaces, we obtain a canonical decomposition $\mathfrak{k} = \mathfrak{k}_p \oplus \mathfrak{n} = \mathfrak{p} \oplus (\mathfrak{m} \oplus \mathfrak{n})$ with the same property: $\mathrm{Ad}(P)(\mathfrak{m} \oplus \mathfrak{n}) \subset \mathfrak{m} \oplus \mathfrak{n}$. By the invariant connection theory [6], there exists on the principal fibre bundle $\pi_{\mathfrak{p}} : K \to \Sigma$ a canonical K-invariant connection because we have a subspace $\mathfrak{m} \oplus \mathfrak{n} \subset \mathfrak{k}$ invariant under the adjoint action of P and such that $\mathfrak{k} = \mathfrak{p} \oplus (\mathfrak{m} \oplus \mathfrak{n})$. Then Proposition 9.1 and Remark 9.2 finish the proof. □

Consider finally the special case $K_p \subset K_f$ ($\Rightarrow Y = Z$). Then, Propositions 8.1 and 9.1 can be used in order to consider the result of Proposition 5.1 from another point of view. Indeed, in that case we have $[\mathfrak{k}, \mathfrak{k}_p] \subset \ker f$, in particular, $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$. We can thus apply Corollary 6.6 and conclude that $\mathfrak{h} = \mathfrak{k}_p^{\mathrm{c}} \oplus_\rho V^{\mathrm{c}}$ is a real polarization of $\mathfrak{g}^{\mathrm{c}}$, satisfying Pukanszky's condition. Now, $P = K_p$ and $\Sigma = Z$. But for the case of real polarizations, the content of Proposition 8.1 is essentially that the coadjoint orbit in question is isomorphic to a modified cotangent bundle $(T^*\Sigma,\ \mathrm{d}\theta_\Sigma + \tau^*\beta_0)$ (see [3, Remark 3.10]). Now, according to Proposition 9.1 and Remark 9.2, the choice of a connection $\alpha \in \Omega^1(K) \otimes \mathfrak{k}_p$ on $K \to K/P$ determines this 2-form completely. The connection α in turn is determined if we fix a subspace $\mathfrak{n} \subset \mathfrak{k}$ such that $\mathfrak{k}_p \oplus \mathfrak{n} = \mathfrak{k}$ and $\mathrm{Ad}(K_p)\mathfrak{n} = \mathfrak{n}$. Then, α is the $\mathfrak{k}_p$-component of the Maurer–Cartan form on K. It must be emphasized here that, the 2-form $\alpha_0 \in \Omega^2(Z)$ appearing in Proposition 5.1 and giving the modification term of the symplectic structure of T^*Z depends on the point (f, p) of the coadjoint orbit for which $K_p \subset K_f$. On the other hand, we have just seen that the 2-form $\beta_0 \in \Omega^2(\Sigma)$ depends on the choice of a connection on the principal bundle $G \to G/D$. Thus the differential forms α_0 and β_0 are not canonical and in general $\alpha_0 \neq \beta_0$. But in any case, Proposition 8.1 tells us that the symplectic structures $\omega_Z + \tau^*\alpha_0$ and $\omega_\Sigma + \tau^*\beta_0$ are equivalent, that is, there exists a bijection $T^*Z \to T^*\Sigma$ which is a symplectomorphism with respect to these structures.

10. Symplectic induction and semidirect product

We have seen previously (Proposition 8.1) that the validity of Pukanszky's condition for a polarization of the coadjoint orbit $\mathcal{O}_\nu^G$ is equivalent to the fact that this orbit is symplectomorphic to a subbundle of a modified cotangent bundle, obtained by symplectic induction from a point. In this section we will discuss a more general property of the coadjoint orbits of a semidirect product. See [8] for an equivalent approach.

We state now the principal result of this section.

Theorem 10.1. *The coadjoint orbit $\mathcal{O}_\nu^G$ through $\nu = (f, p) \in \mathfrak{g}^*$ of a semidirect product $G = K \times_\rho V$ is always obtained by symplectic induction from the coadjoint orbit $\mathcal{O}_{\nu_p}^{G_p}$ of G_p passing through $\nu_p = (i_p^* f, p) \in \mathfrak{g}_p^*$, with groups $G = K \times_\rho V$ and $G_p = K_p \times_\rho V$:*

$$\mathcal{O}_\nu^G = \mathrm{Ind}_{G_p}^G(\mathcal{O}_{\nu_p}^{G_p}).$$

Note that $\mathcal{O}_{\nu_p}^{G_p} = G_p \cdot \nu_p$ *is canonically isomorphic to* $\mathcal{O}_\phi^{K_p} = K_p \cdot \phi$.

Proof. Using the notation of the section on symplectic induction, let us choose the symplectic manifold M as $M = G_p \cdot \nu_p$ and the groups G and H as $G = K \times_\rho V$ and $H = K_p \times_\rho V = G_p$. We will then apply symplectic induction from M with the above mentioned groups.

In our case, one can consider the symplectic manifold M as the coadjoint orbit of K_p passing through ϕ, because $(\kappa, v)\cdot(\phi, p) = (\kappa\cdot\phi + i_p^*(p \odot v), p) = (\kappa\cdot\phi, p), \forall(\kappa, v) \in G_p$, since $i_p^*(p \odot v) = 0$ (see Lemma 2.1). The action of G_p on M is Hamiltonian with momentum mapping $J_M : M \to \mathfrak{g}_p^*$ given by $J_M(m) = m$, $m = (\kappa \cdot \phi, p)$.

Using the conventions of Section 7, one readily verifies that the zero level set of the momentum map $J_{\check{M}} : \check{M} = M \times T^*G \to \mathfrak{h} = \mathfrak{g}_p$ is given by

$$J_{\check{M}}^{-1}(0) = \{((\varphi, p), g, (z, w)) \in M \times T^*G \mid \varphi = i_p^* z,\ w = p,\ g \in G\},$$

and knowing that $\varphi = \kappa \cdot \phi = \kappa \cdot i_p^* f = i_p^*(\kappa \cdot f)$ for some $\kappa \in K_p$, we have the following characterization for $J_{\check{M}}^{-1}(0)$:

$$J_{\check{M}}^{-1}(0) = \{((\kappa\cdot\phi, p), g, (\kappa \cdot f + p \odot v, p)) \mid \kappa \in K_p, g \in G, v \in V\}$$

Then direct calculation shows that the point $((\kappa \cdot \phi, p), g, (\kappa \cdot f + p \odot v, p))$ of $J_{\check{M}}^{-1}(0)$ can be represented in the quotient $J_{\check{M}}^{-1}(0)/G_p$ by the point $(\hat{\kappa} \cdot (\kappa \cdot f + p \odot v) + q \odot \hat{v}, q)$ if $g = (\hat{\kappa}, \hat{v})$, where q represents g in $G/G_p = Z$: $q = \hat{\kappa} \cdot p$. We realize thus easily that the points of the induced manifold $M_{\mathrm{ind}} = \mathrm{Ind}_{G_p}^G(G_p \cdot \nu_p) = J_{\check{M}}^{-1}(0)/G_p$ will be of the form $(\lambda \cdot f + \lambda \cdot p \odot u, \lambda \cdot p)$, $\lambda \in K$, $u \in V$, so M_{ind} and $\mathcal{O}_\nu^G$ are isomorphic as manifolds.

In order to establish a symplectomorphism between M_{ind} and $\mathcal{O}_\nu^G$, we proceed as follows. The left action of G on $\check{M}$, obtained by taking the cotangent lift of the left action of G on itself and letting G act trivially on M, projects on $M_{\mathrm{ind}} \cong \mathcal{O}_\nu^G$ as the coadjoint action of G, as one verifies by easy calculation. We choose now an element $g_0 = (\kappa_0, v_0) \in G$ and let $n_0 = ((\phi, p), (\kappa_0, v_0), (f, p)) \in J_{\check{M}}^{-1}(0)$. The image of n_0 in M_{ind} is equal to $(\kappa_0 \cdot f + \kappa_0 \cdot p \odot v_0, \kappa_0 \cdot p) = (h, q)$. Furthermore, the projection of a vector at n_0 induced by the action of G on $\check{M}$ will coincide with the corresponding vector at $(h, q) \in M_{\mathrm{ind}}$ induced by the coadjoint action of G on M_{ind}. So, consider an element $\xi = (A, a) \in \mathfrak{g}$; then $\xi_{\check{M}}(n_0) = ((0, 0), T_e R_{g_0}(\xi), (0, 0))$ and $[\xi_{\check{M}}(n_0)] = \xi_{\mathfrak{g}^*}(h, q)$. Similarly, if $\eta = (B, b) \in \mathfrak{g}$, then

$$\begin{aligned}(\omega_{\mathrm{ind}})_{(h,q)}(\xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*}) &= -(f, p)([TL_{g_0^{-1}} \circ TR_{g_0}(A, a), TL_{g_0^{-1}} \circ TR_{g_0}(B, b)]) \\ &= -(h, q)([(A, a), (B, b)]).\end{aligned}$$

This shows that ω_{ind} coincides with the standard symplectic structure of the coadjoint orbit $\mathcal{O}_\nu^G$. □

We observe here the following analogy with the construction of Rawnsley [11], described in Section 3. According to Lemma 3.2, we have the fibration $\mathcal{O}_\nu^G \to Y \to Z$, where the typical fibres are, respectively, $p \odot V$ and $\mathcal{O}_{\phi}^{K_p}$. But now Theorems 7.1 and 10.1 ensure that we have indeed a (non-canonical) fibration $\mathcal{O}_{\nu}^{G} \to T^*Z$, with typical fibre $\mathcal{O}_{\phi}^{K_p}$.

Remark 10.2. It is evident that if Z is a contractible space, then the coadjoint orbit $\mathcal{O}_{\nu}^{G}$ is globally diffeomorphic to the product $T^*Z \times \mathcal{O}_{\phi}^{K_p}$. In particular, when K_p is a group-deformation retract of K, that is when there exists a homotopy $H : [0, 1] \times K \to K$ with the properties $H(0, \kappa) = i_p(r(\kappa)), H(1, \kappa) = \kappa$ and $H_t(\kappa\lambda) = H_t(\kappa)H_t(\lambda), H_t(\kappa) = H(t, \kappa)$, then Z is contractible; a homotopy $\check{H} : [0, 1] \times Z \to Z$ between the constant map and the identity on Z, is given by $\check{H}(t, [\kappa]) = [H(t, \kappa)]$.

As an immediate application of Theorem 10.1, we discuss in the light of symplectic induction the result of Proposition 5.1. If $K_p \subset K_f$, then $\mathcal{O}_{\phi}^{K_p} = \{\phi\}$ and the coadjoint orbit of ν is hence obtained by symplectic induction from a point. Thus, according to Proposition 2.11 of [3] det, $\mathcal{O}_{\nu}^{G}$ must be isomorphic to a modified cotangent bundle $T^{\sharp}(G/G_p) = T^{\sharp}Z$, where the modification term is determined by a connection on the principal bundle $G \to G/G_p$, in accordance with Proposition 5.1. The choice of this connection has been discussed in Section 9.

11. Examples

We consider here three representative examples of semidirect product and we apply the general formalism developed in the previous sections. The semidirect product Lie groups we analyse below, are important for the non-relativistic particle dynamics.

11.1. The special Euclidean group of $\mathbb{R}^3$

Let $K = SO(3)$ be the Lie group of rotations in $V = \mathbb{R}^3$ preserving the standard scalar product $\langle -, - \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. The familiar representation of the elements of $SO(3)$ as 3×3 matrices, enables us to form the semidirect product $G = SE(3) = SO(3) \rtimes \mathbb{R}^3$, the Euclidean group in $\mathbb{R}^3$.

The Lie algebra $\mathfrak{se}(3)$ as well as its dual $\mathfrak{se}(3)^*$ are canonically isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^3$. Easy calculation shows that if $p \in V^* \cong \mathbb{R}^3$, then the linear map $\tau_p^* : V = \mathbb{R}^3 \to \mathfrak{k}^* \cong \mathbb{R}^3$ is given by $\tau_p^*(v) = p \times v$ (the usual cross product of the vector space $\mathbb{R}^3$).

Now let $\nu = (f, p) \in \mathfrak{se}(3)^*$ be an element such that $f = s\boldsymbol{u}$, $p = k\boldsymbol{u}$ and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 1$ $(s, k > 0)$. Then $K_p = K_f = (K_p)_\phi \cong SO(2)$, the orbits Z and Q are 2-spheres $\mathbf{S}^2$ and $\mathcal{O}_{\phi}^{K_p} = \{\phi\}$. We can furthermore apply Propositions 4.4 and 5.1 which show that the coadjoint orbit of ν coincides as a manifold to $T^*\mathbf{S}^2$, but its symplectic structure is modified by a "spin term" which, in this case, is s-times the canonical symplectic structure of $\mathbf{S}^2$ (its volume element).

Taking into account the discussion after Proposition 9.3, we can reconsider this result in the context of algebraic polarizations: since $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$ ($\mathfrak{k}_p = \mathfrak{so}(2)$ is an abelian

Lie algebra), Corollary 6.6 can be applied; therefore, the subspace $\mathfrak{a} = \mathfrak{so}(2)^c$ is a real polarization of $\mathfrak{k}_p^c = \mathfrak{so}(2)^c$ satisfying Pukanszky's condition, so

$$\mathfrak{h} = \mathfrak{so}(2)^c \oplus (\mathbb{R}^3)^c$$

is a real polarization of $\mathfrak{g}^c$ (with respect to ν) satisfying also the same condition (see also Proposition 6.3 and Theorem 6.4). Then Proposition 8.1 gives the same result on the structure of $\mathcal{O}_\nu^G$. Alternatively, one could use Theorem 10.1 (see discussion after Remark 10.2).

For this polarization, the groups D and E coincide with $G_p = SO(2) \rhd\!\!< \mathbb{R}^3$. Furthermore, there exists a canonical connection on the principal bundle $G \to G_p$ because the spaces $K_p/P =$ point and $K/K_p = \mathbf{S}^2$ are symmetric spaces (see Proposition 9.3).

11.2. The Galilei group of $\mathbb{R}^3 \oplus \mathbb{R}$

Take now as group K the Lie group $SE(3)$ of the previous example, $K = SO(3) \rhd\!\!< \mathbb{R}^3$ and the vector space V as $\mathbb{R}^3 \oplus \mathbb{R}$ (Galilean space–time). We have a representation $\rho : K \to GL(V)$ given by

$$\rho(R, \boldsymbol{b}) = \begin{pmatrix} R & \boldsymbol{b} \\ 0 & 1 \end{pmatrix}$$

and consequently one can consider the semidirect product $G = K \rhd\!\!< V$. We recognize G as the Galilei group in dimension 3 + 1, see [13].

Using the isomorphism $\mathfrak{g} \cong \mathfrak{g}^* \cong (\mathbb{R}^3 \oplus \mathbb{R}^3) \oplus (\mathbb{R}^3 \oplus \mathbb{R})$, we may represent the elements $f \in \mathfrak{k}^*$ and $p \in V^*$ as $f = (\boldsymbol{l}, \boldsymbol{g})$, $\boldsymbol{l}, \boldsymbol{g} \in \mathbb{R}^3$ and $p = (\boldsymbol{p}, E)$, $\boldsymbol{p} \in \mathbb{R}^3$, $E \in \mathbb{R}$. Under these identifications, if we set $\kappa = (R, \boldsymbol{b}) \in SE(3)$ and $x = \binom{\boldsymbol{r}}{t} \in \mathbb{R}^3 \times \mathbb{R}$, one readily finds:

$$p \odot x = (\boldsymbol{p} \times \boldsymbol{r}, \boldsymbol{p}t) \quad \text{and} \quad \kappa \cdot p = (R\boldsymbol{p}, E - \langle R\boldsymbol{p}, \boldsymbol{b} \rangle). \tag{11.1}$$

(i) Let us choose $\nu = (f, p) \in \mathfrak{g}^*$ as $f = (s\boldsymbol{u}, 0)$, $p = (k\boldsymbol{u}, E)$, $s, k > 0$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 1$. This choice corresponds to the standard non-relativistic particle of mass zero with spin s and colour k, see [13].

By formula (11.1) one easily finds $K_p = SO(2) \rhd\!\!< \mathbb{R}^2$, $\mathbb{R}^2$ being the subspace perpendicular to $\boldsymbol{u}$ and $SO(2)$ the rotation group of this subspace. In this case, only the $\mathfrak{so}(2)^*$-component of $\phi = i_p^* f \in \mathfrak{k}_p^*$ is non-zero and consequently, $\mathcal{O}_\phi^{K_p}$ is a coadjoint orbit of $SO(2)$; so $\mathcal{O}_\phi^{K_p} = \{\phi\}$ because $SO(2)$ is abelian. Furthermore, the homogeneous space $Z = K/K_p$ is simply the product $\mathbf{S}^2 \times \mathbb{R}$. Thus, according to Theorem 10.1 (see also discussion after Remark 10.2), the coadjoint orbit of ν is symplectomorphic to a modified cotangent bundle $T^\sharp(K/K_p) = T^\sharp\mathbf{S}^2 \times \mathbb{R}^2$.

One can obtain the same result using the technique of polarizations. Indeed, with the previous choices, we have $[f] \in \mathrm{H}^1(\mathfrak{k}_p, \mathbb{R})$ (because $SO(2)$ is abelian) and so Corollary 6.6 can be applied. The real polarization $\mathfrak{h}$ provided by this corollary is

$$\mathfrak{h} = (\mathfrak{so}(2)^c \oplus (\mathbb{R}^2)^c) \oplus (\mathbb{R}^3 \oplus \mathbb{R})^c$$

and the groups D and E

$$D = E = (SO(2) \rtimes \mathbb{R}^2) \rtimes (\mathbb{R}^3 \times \mathbb{R}).$$

Then, according to Proposition 8.2, the coadjoint orbit $\mathcal{O}_\nu^G$ is symplectomorphic to a modified cotangent bundle $T^\sharp(K/K_p) = T^\sharp \mathbf{S}^2 \times \mathbb{R}^2$.

(ii) We choose now an element $\nu = (f, p) \in \mathfrak{g}^*$ setting $p = (\boldsymbol{p}, 0)$, $\langle \boldsymbol{p}, \boldsymbol{p} \rangle = 1$ and $f = (0, \boldsymbol{g})$ with $\langle \boldsymbol{g}, \boldsymbol{p} \rangle = 0$ and $\langle \boldsymbol{g}, \boldsymbol{g} \rangle = 1$. The coadjoint orbit of ν has a less evident interpretation; according to [4], it corresponds to "particles at infinity with infinite velocity and mass zero".

Now, by Eq. (11.1) we find $K_p = SO(2) \rtimes \mathbb{R}^2$, where $\mathbb{R}^2$ is the two-dimensional subspace of $\mathbb{R}^3$ perpendicular to the line $\mathbb{R}\boldsymbol{p}$ and $SO(2)$ the special orthogonal group of this subspace. On the other hand, the projection $\phi = i_p^* f$ is equal to f and we readily obtain $(K_p)_\phi = \{e\} \rtimes \mathbb{R}\boldsymbol{g}$. Furthermore, $\ker \tau_p^* = \mathbb{R}\boldsymbol{p}$, as one can see from (11.1). Thus, the isotropy subgroup G_ν is two-dimensional and the orbit $\mathcal{O}_\nu^G$ will be eight-dimensional. Indeed, as in the previous example, the orbit $Z = K/K_p$ is the product $\mathbf{S}^2 \times \mathbb{R}$ and by Theorem 10.1, the coadjoint orbit of ν can be identified (in a non-canonical way) with a fibre bundle over $T^*Z = T^*\mathbf{S}^2 \times \mathbb{R}^2$ whose typical fibre is $\mathcal{O}_\phi^{K_p} = K_p/(K_p)_\phi \cong T^*\mathbf{S}^1 \cong \mathbf{S}^1 \times \mathbb{R}$.

However, we can further investigate the structure of this orbit as follows. Observe first that the subspace $\mathfrak{a} = (0 \oplus \mathbb{R}^2)^c \subset \mathfrak{k}_p^c$ is a real polarization with respect to ϕ. In fact, $(\mathfrak{k}_p)_\phi^c \subset \mathfrak{a}$, $\mathfrak{a}$ is invariant under the adjoint action of $(K_p)_\phi$ (see relation (2.4)), $\dim_{\mathbb{C}} \mathfrak{a} = \frac{1}{2}(\dim \mathfrak{k}_p + \dim (\mathfrak{k}_p)_\phi)$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. In this case $\mathfrak{d} = \mathfrak{e} = 0 \oplus \mathbb{R}^2$ and $D = E = \{e\} \rtimes \mathbb{R}^2$. As a result, $D \cdot \phi = (\mathbb{R}, \boldsymbol{g}) = \phi + \mathfrak{e}^\circ$, which means that $\mathfrak{a}$ satisfies Pukanszky's condition. By Theorem 6.4, the subspace

$$\mathfrak{h} = \mathfrak{a} \oplus V^c = (0 \oplus \mathbb{R}^2)^c \oplus (\mathbb{R}^3 \oplus \mathbb{R})^c \subset \mathfrak{g}^c$$

is a real polarization of $\mathfrak{g}^c$ (with respect to ν) satisfying also Pukanszky's condition. Then Proposition 8.2, applied for a real polarization, tells us that the coadjoint orbit $\mathcal{O}_\nu^G$ is symplectomorphic to a modified cotangent bundle $T^\sharp(G/D) \cong T^\sharp(SO(3) \times \mathbb{R})$. In particular, $\mathcal{O}_\nu^G \cong SO(3) \times \mathbb{R}^5$ as a manifold.

11.3. The Bargmann group of $\mathbb{R}^3 \oplus \mathbb{R}$

Consider again the special euclidean group $SE(3)$ of $\mathbb{R}^3$ and let $\rho : SE(3) \to GL(\mathbb{R}^5)$ be the representation given by

$$\rho(R, \boldsymbol{b}) = \begin{pmatrix} R & \boldsymbol{b} & 0 \\ 0 & 1 & 0 \\ -\boldsymbol{b}^t R & -\boldsymbol{b}^2/2 & 1 \end{pmatrix}.$$

The semidirect product $G = SE(3) \times_\rho \mathbb{R}^5$ is called Bargmann group and it is a non-trivial extension of the Galilei group, previously studied, by $\mathbb{R}$, see [1]. If we write an element $p \in \mathbb{R}^5$ as $p = (\boldsymbol{p}, E, m)$ and $\kappa = (R, \boldsymbol{b}) \in K = SE(3)$, we find easily

$$\kappa \cdot p = \left(Rp + mb,\, E - \langle Rp, b\rangle - m\frac{b^2}{2},\, m \right). \tag{11.2}$$

Let us consider the coadjoint orbit of the element $\nu = (f, p) \in \mathfrak{g}^*$ with $f = (s\boldsymbol{u}, 0)$ and $p = (0, 0, m)$, $m > 0$, s and $\boldsymbol{u}$ as above. Easy calculation using (11.2) shows that $K_p = SO(3) \rhd\!\!< \{0\}$ and consequently the projection $i_p^* : \mathfrak{k}^* \to \mathfrak{k}_p^*$ is simply the projection on to the first factor, $i_p^* f = \phi = s\boldsymbol{u}$. Now the orbit $Z = K/K_p$ is simply $Z = \mathbb{R}^3$ and $\mathcal{O}_\phi^{K_p} \cong \mathbf{S}^2$, thus by Remark 10.2 the coadjoint orbit $\mathcal{O}_\nu^G$ of ν is diffeomorphic to the product $T^*\mathbb{R}^3 \times \mathbf{S}^2$. We recognize here the phase space of non-relativistic particles of mass m and spin s [13].

One could also investigate the structure of $\mathcal{O}_\nu^G$ using the technique of algebraic polarizations. To this end, one proceeds as follows. Consider the subspaces

$$\mathfrak{a}_0^{\pm} = \{A \in \mathfrak{k}_p^{\mathrm{c}} \mid \hat{A} \times u = \pm \mathrm{i}\hat{A}\}$$

of $\mathfrak{k}_p^{\mathrm{c}} = \mathfrak{so}(3)^{\mathrm{c}}$, where $A \mapsto \hat{A}$ is the natural isomorphism $\mathfrak{so}(3)^{\mathrm{c}} \cong (\mathbb{R}^3)^{\mathrm{c}}$. These subspaces have complex dimension 1 and are such that $\mathfrak{a}_0^+ \oplus \mathfrak{a}_0^- \oplus (\mathfrak{k}_p)_\phi^{\mathrm{c}} = \mathfrak{k}_p^{\mathrm{c}}$. Furthermore, it is elementary to verify that $[\mathfrak{a}_0^+, \mathfrak{a}_0^+] \subset (\mathfrak{k}_p)_\phi^{\mathrm{c}}$ and $[\mathfrak{a}_0^+, (\mathfrak{k}_p)_\phi^{\mathrm{c}}] \subset \mathfrak{a}_0^+$ (similarly for $\mathfrak{a}_0^-$). Thus, if we set

$$\mathfrak{a}^{\pm} = \mathfrak{a}_0^{\pm} \oplus (\mathfrak{k}_p)_\phi^{\mathrm{c}},$$

we obtain two (isomorphic) complex subalgebras of $\mathfrak{k}_p^{\mathrm{c}}$ such that complex conjugation on the one yields the other. This means that $\mathfrak{a}^{\pm}$ are (isomorphic) complex polarizations of $\mathfrak{k}_p^{\mathrm{c}}$ with respect to $\phi = s\boldsymbol{u}$. The real Lie subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ of $\mathfrak{k}_p$ are easily found to be $\mathfrak{d} = (\mathfrak{k}_p)_\phi = \mathfrak{so}(2)$ and $\mathfrak{e} = \mathfrak{k}_p = \mathfrak{so}(3)$. It is then evident that $D \cdot \phi = \phi$ and so $\mathfrak{a}^{\pm}$ satisfy Pukanszky's condition because $\mathfrak{e}^\circ = 0$. As a result, the subalgebras

$$\mathfrak{h}^{\pm} = \mathfrak{a}^{\pm} \oplus_\rho (\mathbb{R}^5)^{\mathrm{c}}$$

of $\mathfrak{g}^{\mathrm{c}}$ are complex polarizations satisfying Pukanszky's condition. Therefore, by Proposition 8.2 we conclude that the coadjoint orbit $\mathcal{O}_\nu^G$ is a fibre bundle over $T^*(G/E) = T^*\mathbb{R}^3$ with typical fibre $E/D = \mathbf{S}^2$.

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Note added in proof

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